A STUDY OF UNIVALENCE OF GELFOND - LEONTEV DERIVATIVES OF ANALYTIC FUNCTIONS AND RELATED FUNCTION CLASSES

by

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A STUDY OF UNIVALENCE OF GELFOND - LEONTEV DERIVATIVES OF ANALYTIC FUNCTIONS AND RELATED FUNCTION CLASSES

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In

Loving Memory of

My Grandfather

CERTIFICATE

This is to certify that the work embodied in the thesis entitled A STUDY OF UNIVALENCE OF GELFOND-LEONTEV DERIVATIVES OF ANALYTIC FUNCTIONS AND RELATED FUNCTION CLASSES being submitted by Jagannath Patel has been carried out under my supervision and that it has not been submitted elsewhere for the award of any degree or diploma.

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CONTENTS

CHAPTER I

INTRODUCTION

1.1 A complex valued function is said to be analytic in a domain Ω (a non-empty open connected subset of the complex plane $\mathbb{C}$) if it has a uniquely determined derivative at each point of Ω . A function f is said to be univalent in a domain Ω if it never takes any value more than once , that is , the condition $f(z_1) = f(z_2)$, $z_1, z_2 \in \Omega$, implies $z_1 = z_2$. A necessary condition for an analytic function $f(z)$ to be univalent in Ω is $f'(z) \neq o$ in Ω . That this condition is not sufficient can be seen by considering the function $f(z) = e^z$ whose derivative never vanishes but clearly it is not univalent in $\mathbb{C}$.

In the study of univalent analytic functions , the Riemann mapping theorem plays an important role . The theorem states that if Ω is a simply connected domain whose boundary consists of more than one point and z_o is a given point in Ω then there exists a unique univalent analytic function f which maps Ω conformally onto the unit disc $\Delta = \{ z: |z| < 1 \}$ and has the properties $f(z_o) = o$, $f'(z_o) = 1$ .Thus , for the study of geometric properties of functions univalent and analytic in a simply connected domain with more than one boundary point one may therefore confine , without loss of generality , to functions univalent and analytic in the unit disc Δ .

If $g(z)$ is univalent and analytic in Δ, so is the function $f(z) = (g(z)-g(o))/g'(o)$ since $g'(o) \neq o$. Thus, it is enough to consider univalent analytic functions in Δ satisfying $f(o) = o$, $f'(o) = 1$. Let H denote the class of functions f analytic in Δ and normalized by the conditions $f(o) = o$, $f'(o) = 1$ and let S be the class of functions f in H that are univalent in Δ. The Taylor series expansion of such a function f about the origin has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1.1.1}$$

Unless otherwise stated explicitly it is assumed throughout in the sequel that whenever $f(z)$ is in S, it has Taylor series representation of the form (1.1.1).

The study of theory of univalent functions was initiated by Koebe [47] in 1907 on the uniformization of algebric curves. He proved that the ranges of all functions in S contain a common disc $|w| < b$, where b is an absolute constant. The Koebe function $k(z) = z(1-z)^{-2}$ shows that $b \leq \frac{1}{4}$. Bieberbach [3] established that $b = \frac{1}{4}$. He also proved in the same paper that if $f \in S$, then $|a_2| \leq 2$ with equality occuring only for the rotations $e^{-i\theta}k(e^{i\theta}z)$ of the Koebe function $k(z)$. Motivated by these extremal properties of the Koebe function, Bieberbach conjectured that for every $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$,

$$|a_n| \leq n , \quad n = 2, 3, \ldots . \tag{1.1.2}$$

Equality occurs in (1.1.2) for each n, if and only if $f(z)$ is the Koebe function $k(z)$ or one of its rotations.

A conjecture stronger than Bieberbach conjecture is due to Robertson [70] which asserts that if $f(z) \in S$, then the coefficients of odd univalent functions

(1.1.3) $$h(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots$$

satisfy the inequality

(1.1.4) $$\sum_{k=1}^{n} |c_{2k-1}|^2 \leq n \ , \ n = 1, 2, \ldots \ .$$

where $c_1 = 1$.

In 1971 , Milin [56] proposed the following conjecture : If $f(z) \in S$, then

(1.1.5) $$\sum_{m=1}^{n} \sum_{k=1}^{m} (k|\gamma_k|^2 - \frac{1}{k}) \leq 0 \ , \ n = 1, 2, \ldots \ .$$

where γ_n , $n = 1, 2, \ldots,$ are given by

(1.1.6) $$\log(\frac{f(z)}{z}) = 2 \sum_{n=1}^{\infty} \gamma_n z^n$$

Milin conjecture is strongest in the sense that it implies Robertson conjecture and hence Bieberbach conjecture .

Recently , Milin's conjecture has been proved by Louis de Branges [22] . With this , both Bieberbach conjecture and Robertson conjecture stand proved in affirmative . A different and simplifed version of Louis de Brange's proof has now been given by Fitzgerald and Pommerenke [25] . During the process of solution of the Bieberbach and related conjectures , several subclasses of univalent functions , important on their own right , were introduced and different techniques like Lowner parametric method , convolution techniques,

variational method , extreme point theory etc. were discovered. All these developments are amply reflected in Bernardi's Bibliography of Univalent Function [1]. The texts of Goluzin [28] , Jenkins [38] , Pommerenke [64], Schober [75], Goodman [30,31] and Duren [23] cover almost all the fundamental aspects of the theory of univalent functions .

Wherever there is no ambiguity , we will use the term univalent functions in the sequel for analytic univalent functions.

1.2 In this section we give some definitions and basic results concerning the class S and some of the subclasses of S that are needed in the sequel.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in S. Then [30]

(1.2.1) $$|a_2^2 - a_3| \leq 1$$

Further , if $|f(z)| < M$ for $z \in \Delta$ then [62]

(1.2.2) $$|a_2| \leq 2(1 - \frac{1}{M}) \; , \; M > 1$$

and

(1.2.3) $$|a_3| \leq \begin{cases} 1 - \dfrac{1}{M^2} \; , \; 1 \leq M \leq e \\ 2(\sigma - \dfrac{1}{M^2}) + 1 - \dfrac{1}{M^2} \; , \; e \leq M \end{cases}$$

where σ is the real root of $Mx.\log x = -1$ [103].

Brannan [10] , Rahman and Szynal [66] , Suffridge [102] and others obtained bounds for the coefficients of certain polynomials belonging to the class S. For instance , Brannan proved that the function $g(z) = z + b_2 z^2 + b_3 z^3$, b_2 and b_3 real , is in S , if and only if ,

$$(1.2.4) \qquad |b_2| \le \begin{cases} \dfrac{1+3b_3}{2}, & -\frac{1}{3} \le b_3 \le \frac{1}{5} \\ 2\sqrt{b_3(1-b_3)}, & \frac{1}{5} \le b_3 \le \frac{1}{3} \end{cases}$$

Rahman and Szynal [66] showed that a polynomial $g(z) = z+b_3z^3+b_5z^5$, b_3 is real, b_5 is real and positive, is in S, if and only if,

$$(1.2.5) \qquad |b_3| \le \begin{cases} \dfrac{1+5b_3}{3}, & 0 \le b_5 \le \frac{1}{10} \\ 2\sqrt{b_5(1-b_5)}-b_5, & \frac{1}{10} \le b_5 \le \frac{1}{5} \end{cases}$$

The coefficient estimate of polynomials of degree n with fixed nth coefficient is obtained by Suffridge [102] who proved that if $g(z) = z + \sum_{k=2}^{n} a_k z^k$, a_k's are real and $|a_n| = \frac{1}{n}$, then

$$(1.2.6) \qquad |a_k| \le A_{k,1}$$

where

$$A_{k,1} = \frac{n-k+1}{n} \frac{\sin(\frac{k\pi}{n+1})}{\sin(\frac{\pi}{n+1})}, \quad k = 1, 2, \ldots, n.$$

The inequality (1.2.6) is sharp, with equality for the polynomial $P(z) \equiv P(z,n,1) = \sum_{k=1}^{n} A_{k,1} z^k$.

A domain Ω in the complex plane is said to be convex if it contains the line segment joining z_1 and z_2 for all distinct

points z_1 and z_2 in Ω .

Definition 1.2.1 A function $f \in H$ is said to be convex if f maps Δ onto a convex domain.

We denote the class of convex functions by K . An analytic characterization of f in K is due to Robertson [69]. Thus , a function f in H is in K , if and only if , for $z \in \Delta$

(1.2.7) $$\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} > 0$$

A function $f \in H$ is said to be convex of order $\alpha (0 \leqq \alpha \leqq 1)$, if , for $z \in \Delta$

(1.2.8) $$\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geq \alpha \ .$$

Denote the class of convex functions of order α by $K(\alpha)$. It is easily seen that $K(0) \equiv K$, $K(\alpha) \subset K$, $0 \leqq \alpha \leqq 1$ and $K(1) = \{z\}$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then it is known [30] that a necessary condition for f to be in K is

(1.2.9) $$|a_n| \leqq 1 \ , \ n = 2, 3, \ldots$$

with equality occuring for the function $f(z) = z(1-z)^{-1}$.

A sufficient condition for f to be in K is that [30]

(1.2.10) $$\sum_{n=2}^{\infty} n^2 |a_n| \leqq 1 \ .$$

Analogous to (1.2.1) , if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in K , then Trimble [106] proved that

(1.2.11) $$|a_2^2 - a_3| \leqq \frac{1}{3}(1 - |a_2|^2)$$

The bound in (1.2.11) is an improvement on the bound of $|a^2 - a_3|$ for $f \in K$ obtained earlier in [46] and [71].

For the polynomial $g(z) = z+b_2z^2+b_3z^3$, b_3 real and positive, Suffridge [102] proved that $g(z)$ is in K, if and only if,

$$(1.2.12) \qquad |b_2| \le \begin{cases} \dfrac{1+9b_3}{4}, & 0 \le b_3 \le \dfrac{1}{15} \\ 2\left[\dfrac{2b_3(1-9b_3)}{3-25b_3}\right]^{\frac{1}{2}}, & \dfrac{1}{15} \le b_3 \le \dfrac{1}{9} \end{cases}$$

A domain Ω in the complex plane is said to be starlike with respect to the point $w_0 \in \Omega$ if the line segment joining w_0 to every other point $w \in \Omega$ lies entirely in Ω.

<u>Definition</u> 1.2.2 A function $f \in H$ is said to be <u>starlike</u> with respect to the point w_0 if f maps Δ onto a domain that is starlike with respect to the point w_0.

The class of starlike functions with respect to origin is denoted by S^*.

It is observed that $K \subset S^*$. The containment is proper since the Koebe function $k(z) = z(1-z)^{-2}$ is in S^* but is not in K. Robertson [69] proved that a function $f \in H$ is in S^*, if and only if, for $z \in \Delta$

$$(1.2.13) \qquad \mathrm{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > 0$$

A function $f \in H$ is said to be starlike of order α $(0 \le \alpha \le 1)$, if for $z \in \Delta$,

$$(1.2.14) \qquad \mathrm{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \ge \alpha .$$

Denote by $S^*(\alpha)$, the class of starlike functions of order α.

It follows that $S^*(o) \equiv S^*$, $S^*(\alpha) \subset S^*$, $o \leq \alpha \leq 1$ and $S^*(1) = \{z\}$. The inequalities (1.2.8) and (1.2.14) reveal a close connection between convex and starlike functions; a function $f \in K(\alpha)$ if and only if $zf' \in S^*(\alpha)$.

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then a necessary condition for f to be in S^* is that [30]

(1.2.15) $$|a_n| \leq n, \quad n = 2, 3, \dots$$

The inequality in (1.2.15) is sharp for the Koebe function $k(z) = z(1-z)^{-2}$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be defined in Δ. If

(1.2.16) $$\sum_{n=2}^{\infty} n|a_n| \leq 1$$

then $f(z)$ is in S^*. If (1.2.16) is replaced by

(1.2.17) $$\sum_{n=2}^{\infty} \frac{(n+k)!}{(n-1)!} |a_{n+k}| \leq |a_{k+1}|(k+1)!$$

and if $a_{k+1} \neq o$, then $f^{(k)}(z)$ is in S^* for $k = 1, 2, 3, \dots$

A natural generalization of starlike functions is the class of spiral-like functions:

Definition 1.2.3 A function f in S is called λ-spirallike, $-\pi/2 < \lambda < \pi/2$, if for each w in $f(\Delta)$ and $t \geq o$ the logarithmic spiral $w\exp(e^{-i\lambda}t)$ is contained in $f(\Delta)$.

We denote the class of λ-spirallike functions for a specific value of λ by $S(\lambda)$. It is known [100] that a function $f \in H$ is λ-spirallike, $-\pi/2 < \lambda < \pi/2$, if and only if, for $z \in \Delta$

$$\text{(1.2.18)} \qquad \operatorname{Re}\left\{e^{i\lambda}\, z\, \frac{f'(z)}{f(z)}\right\} > 0 .$$

Clearly, $S(0) = S^*$.

Libera [50] introduced the class of λ-spirallike functions of order $\alpha (0 \leqq \alpha \leqq 1)$, denoted by $S(\lambda, \alpha)$, consisting of all functions $f \in H$ that satisfy

$$\text{(1.2.19)} \qquad \operatorname{Re}\left\{e^{i\lambda}\, z\, \frac{f'(z)}{f(z)}\right\} > \alpha\cos\lambda$$

for $z \in \Delta$, $-\pi/2 < \lambda < \pi/2$ and $0 \leqq \alpha \leqq 1$. Obviously, $S(\lambda, 0) \equiv S(\lambda)$. Libera [50] proved that $f \in S(\lambda, \alpha)$ is univalent and, alongwith several other results, he found coefficient bounds for such functions. Thus, it is shown [50] that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\lambda, \alpha)$, then

$$|a_n| \leqq \prod_{k=0}^{n-2} \frac{|2(1-\alpha)\cos\lambda e^{-i\lambda} + k|}{k+1}, \quad n = 2, 3, \ldots$$

and that equality occurs for the function $f(z) = z(1-z)^{-2(1-\alpha)\cos e^{-i\lambda}}$.

For a fixed positive integer N, let $\mu_0 = \exp(2\pi i/N)$ be the Nth root of unity. For a function $f(z) = z + a_2 z^2 + \ldots$, define

$$\text{(1.2.20)} \qquad f_N(z) = \frac{1}{N} \sum_{j=0}^{N-1} \mu_0^{-j} f(\mu_0^j z)$$

$$= z + a_{N+1} z^{N+1} + a_{2N+1} z^{2N+1} + \ldots$$

where we use the fact that

$$\sum_{j=0}^{N-1} \mu_0^{jk} = \begin{cases} N, \text{if } k \text{ is a multiple of } N \\ 0, \text{ otherwise.} \end{cases}$$

Definition 1.2.4 A function $f \in H$ is said to be λ-spirallike of order α with respect to N-symmetric points $(|\lambda|< \pi/2, o\leq \alpha<1)$, if for $z \in \Delta$

$$\text{(1.2.21)} \qquad \operatorname{Re}\left\{e^{i\lambda} \; z \frac{f'(z)}{f_N(z)}\right\} > \alpha\cos\lambda$$

where, $f_N(z)$ is given by (1.2.20).

Denote by $S^N(\lambda,\alpha)$ the class of λ-spirallike functions of order α with respect to N-symmetric points. Clearly, $S^N(\lambda,\alpha)$ is contained in $S(\lambda,\alpha)$. For $\lambda = o$, the particular class $S^N(o,\alpha) \equiv S^{*N}(\alpha)$, the class of starlike functions of order α with respect to N-symmetric points, was studied earlier in ([20],[99]).

Replacing z by $\mu_o z$, $\mu_o^2 z$, ..., $\mu_o^{N-1} z$ in (1.2.21) and adding the N equations so obtained we have, for $z \in \Delta$,

$$\operatorname{Re}\left\{ e^{i\lambda} \; z \frac{f_N'(z)}{f_N(z)}\right\} > \alpha\cos\lambda .$$

This implies that $f_N(z)$ is in $S(\lambda,\alpha)$ and hence the function $f_N(z)$ is univalent.

Silverman and Silvia [95,96] studied the effect of the moduli of second coefficient for functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ on the growth of other coefficients for convex, starlike and other subclasses of univalent functions. Extending some of these results, the influence of the second coefficient on the growth of the other coefficients for functions belonging to the classes $S(\lambda,\alpha)$, and $S^N(\lambda,\alpha)$ are found

in [67]. Some other problems for functions with fixed second coefficient ~~is~~ are studied in [24] , [34] , [42] , [97] and [104] etc.

Another important subclass of λ-spirallike functions studied recently by Mogra and Ahuja [58] is the following :

<u>Definition</u> 1.2.5 A function $f \in H$ is said to be <u>λ-spirallike function of order</u> α <u>and type</u> β , denoted by $S(\lambda, \alpha, \beta)$, if and only if , for $z \in \Delta$

$$(1.2.22) \qquad \left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\beta\left(\frac{zf'(z)}{f(z)} - 1 + (1-\alpha)\cos\lambda e^{-i\lambda}\right) - \left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < 1$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $|\lambda| < \pi/2$. Since $S(\lambda, \alpha, \beta) \subset S(\lambda)$, it follows that the functions in $S(\lambda, \alpha, \beta)$, are univalent . For different values of the parameter λ, α , and β, $S(\lambda, \alpha, \beta)$ reduces to several known subclasses of univalent functions ([27],[40], [50],[53]). If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\lambda, \alpha, \beta)$, then a bound on the coefficients a_n is found in [58] besides several other results .

MacGregor [52] obtained upper bounds for the moduli of the coefficients of a starlike function whose power series representation in Δ is of the form

$$(1.2.23) \qquad f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$$

for some $k = 1, 2, \ldots$. Boyd [9] and Srivastava [101] extended MacGregor's result respectively to the classes S^* and

$S(\lambda,\alpha)$. An analogous result for functions in $S(o,\alpha,\beta)$ is obtained by Mogra and Juneja [57].

The functions in the class S whose non-zero coefficients , from the second on , are negative have many interesting properties . Let T denote the class of functions of the form

(1.2.24) $$f(z) = z - \sum_{n=2}^{\infty} a_n z^n , \; a_n \geq o$$

that are analytic and univalent in the unit disc Δ and let C denote the subclass of T which are convex in Δ . These classes and their subclasses have been studied extensively by Silverman [92] , Silverman and Silvia [95] , Gupta and Jain [32,33] , Kapoor and Mishra [43] and others. It is proved in [92] that, if $f \in T$, then

(1.2.25) $$a_n \leq \frac{1}{n} , \; n = 2,3,\ldots$$

holds and the inequality in (1.2.25) is sharp . A necessary and sufficient condition for a function f , given by (1.2.24) , to be in T is that

(1.2.26) $$\sum_{n=2}^{\infty} n a_n \leq 1 .$$

In view of (1.2.16) and (1.2.26) the functions in T are starlike , i.e., T is contained in S^*. Likewise , a necessary condition for a function f given by (1.2.24) , to be in C is that [92]

(1.2.27) $$a_n \leq \frac{1}{n^2}$$

and the inequality (1.2.27) is sharp . A necessary and sufficient condition for f , given by (1.2.24) in T to be in C is that

(1.2.28) $$\sum_{n=2}^{\infty} n^2 a_n \leq 1.$$

The functions f belonging to certain subclasses of T defined by taking into account the univalence of derivatives of f also have elegant properties . Some such classes have recently been introduced and investigated by Silverman [94] . Thus , let T_1 denote the class of functions f in T for which f' is also univalent in Δ . The class of functions f in T for which the first m derivatives of f are univalent in Δ is denoted by T_m . If $f \in T_m$ for every m , then f is said to be in T_∞.

It is known that if f , given by (1.2.24) , is in T , then the sharp inequality

(1.2.29) $$a_2 \leq \frac{1}{2}$$

holds . Silverman further showed that the sharp inequality (1.2.29) continues to hold even for functions belonging to the subclass T_1 of T . However , he proved [94] that the bound for the third coefficient $a_3 \leq \frac{1}{3}$ for functions in T can be improved as

(1.2.30) $$a_3 < \frac{1}{6}$$

if f , given by (1.2.24) , is in T_1 . In the same paper , he proved that if f , given by (1.2.4), is in T and $a_2 > 0$ then a sufficient condition for f to be in T_1 is that

(1.2.31) $$\sum_{n=3}^{\infty} (n-1) n a_n \leq 2a_2 .$$

If $o < a_2 < \frac{1}{3}$, then (1.2.31) is sufficient for any function f , given by (1.2.24) , to be in T_1 .

For a cubic polynomial $g(z) = z-b_2z^2-b_3z^3$, b_2, b_3 real , in T , (1.2.31) is both necessary and sufficient condition for g(z) to be in T_1 . In fact , it is known [94] that a necessary and sufficient condition for g(z) to be in T_1 is that

$$(1.2.32) \quad 3b_3 \leq \min \begin{cases} 1-2b_2 \\ b_2 \end{cases}$$

Using (1.2.32) , the sharp inequality

$$(1.2.33) \qquad b_3 \leq \frac{1}{9}$$

follows [94] . From (1.2.32) and (1.3.33) , one may be tempted to think that (1.2.31) is both a necessary and sufficient condition for a function f in T_1 and 1/9 is the sharp bound for the third coefficient for any function f , given by (1.2.24). However , none of these are true even for quadratic polynomials in T_1 [94]. In fact , if $g(z) = z-b_2z^2-b_3z^3-b_4z^4 \in T_1$, then the sharp inequality

$$(1.2.34) \qquad b_3 \leq \frac{\sqrt{2}-1}{3}$$

holds . A combination of (1.2.30) and (1.2.34) yields , if $\beta_1 = \sup\{a_3 : f \in T_1\}$, then

$$(1.2.35) \qquad \frac{\sqrt{2}-1}{3} \leq \beta_1 < \frac{1}{6}$$

A sufficient condition for functions in T whose derivatives of higher order are univalent in Δ has also been found [94]

Thus , if $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ with $\prod_{n=2}^{m+1} a_n \neq 0$ and if

$$\sum_{n=k+2}^{\infty} (n-k)(n-k+1)\ldots n a_n \leq (k+1)!\, a_{k+1} \tag{1.2.36}$$

for $k = 1, 2, \ldots, m$ then , $f \in T_m$.

If (1.2.36) is true for all k , then $f \in T_\infty$. The second coefficient of a function f in T satisfies $a_2 < \frac{1}{2}$, and surprisingly , there is no extremal function in T_∞ for this inequality [94].

1.3 Let $f(z)$ be an entire function of a complex variable z i.e., f is analytic for every $z = re^{i\theta}$ in the complex plane $\mathbb{C}$. Set ,

$$M(r) \equiv M(r, f) = \max_{|z|=r} |f(z)| .$$

The function $M(r)$ is called the maximum modulus of $f(z)$ for $|z| = r$. Blumenthal [4] has shown that $M(r, f)$ is a steadily increasing function of r and that it is differentiable in adjacent intervals. Further, by Hadamard's three circle's theorem , it follows that $\log M(r, f)$ is a convex function of $\log r$.

An entire function $f(z)$ is said to be of finite order if there exists a constant A such that

$$M(r, f) < \exp(r^A) \tag{1.3.1}$$

for sufficiently large values of r . The greatest lower bound Λ of all such numbers A is called the <u>order</u> of the entire function $f(z)$. Thus ,

$$\Lambda = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} \tag{1.3.2}$$

If no constant A can be found such that (1.3.1) holds, then $f(z)$ is said to be of infinite order and such functions are said to be of fast growth. The entire functions of zero order are said to be of slow growth. A constant is of zero order by convention.

For a more precise specification of the rate of growth of $f(z)$, the concept of type has been introduced. An entire function $f(z)$ having finite order Λ $(0 < \Lambda < \infty)$ is said to be of finite type, if there exists a constant B such that

$$M(r, f) < \exp(Br^{\Lambda}) \tag{1.3.3}$$

for sufficiently large values of r. The greatest lower bound τ of all such numbers B is called the <u>type</u> of the entire function $f(z)$. Thus,

$$\tau = \limsup_{r \to \infty} \frac{\log M(r)}{r^{\Lambda}} \tag{1.3.4}$$

If no constant B can be found such that (1.3.3) holds, then $f(z)$ is said to be of infinite type.

An entire function $f(z)$ is said to have growth (Λ, τ) if its order does not exceed Λ and its type does not exceed τ if it is of order Λ. An entire function of growth $(1, \tau)$, $\tau < \infty$ is called a function of <u>exponential type</u>.

In 1933, Whittaker [107] introduced the concept of <u>lower order</u> of an entire function. Thus, an entire function $f(z)$ is said to be of lower order Λ_* $(0 < \Lambda_* < \infty)$ if

$$\Lambda_* = \liminf_{r \to \infty} \frac{\log \log M(r)}{\log r} . \tag{1.3.5}$$

An entire function $f(z)$ is said to be of regular growth if $\Lambda = \Lambda_*$ and is said to be of irregular growth if $\Lambda_* < \Lambda$

In analogy with lower order, Shah [78] introduced the concept of lower type. Thus, an entire function $f(z)$ having order Λ $(0 < \Lambda < \infty)$ is said to be of <u>lower type</u> τ_*, if

(1.3.6) $$\tau_* = \liminf_{r \to \infty} \frac{\log M(r)}{r^{\Lambda}}.$$

Since the entire function $f(z)$ is analytic everywhere in the complex plane $\mathbb{C}$, it can be expanded in a Taylor series around any point $z_0 \in \mathbb{C}$. However, without loss of generality, we may take $z_0 = 0$. Then, $f(z)$ has the representation

(1.3.7) $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where the coefficients a_n's are given by

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz = \frac{f^{(n)}(0)}{n!}$$

$f^{(n)}(0)$ being the value of the nth derivative of the function $f(z)$ at $z = 0$. For the entire function $f(z)$, given by (1.3.7), the characterization for finite order Λ and finite type τ, in terms of the Taylor coefficients have been found [6, pp 9-11] Thus, an entire function f, given by (1.3.7), is of finite order, if and only if,

(1.3.8) $$\mu_1 = \limsup_{n \to \infty} \frac{n \log n}{-\log |a_n|}$$

is finite; and the order Λ of $f(z)$ is equal to μ_1. If $\mu_1 = \infty$, then $f(z)$ is said to have of infinite order or else $f(z)$ is not entire.

If $f(z)$ is of finite order $\Lambda\ (0 < \Lambda < \infty)$, then the type $\tau\ (0 < \tau < \infty)$ of $f(z)$ is given by

$$(1.3.9) \qquad e\Lambda\tau = \limsup_{n\to\infty} n|a_n|^{\Lambda/n}.$$

When $\Lambda = 1$, (1.3.9) becomes

$$(1.3.10) \qquad e\tau = \limsup_{n\to\infty} n|a_n|^{1/n}.$$

Since the power series (1.3.7) converges absolutely for all finite z, $|a_n|r^n \to 0$ as $n\to\infty$ for every finite r. Hence, there is one term of the series whose absolute value is not less than that of any other term. The modulus of this term, denoted by $\mu(r, f)$, is called the maximum term of $f(z)$ for $|z| = r$. Thus,

$$\mu(r) \equiv \mu(r, f) = \max_{n\geq 0} \{|a_n|r^n\}.$$

Let

$$\nu(r) \equiv \nu(r, f) = \max\{n : \mu(r) = |a_n|r^n\}.$$

Then $\nu(r)$ is called the central index of $f(z)$ for $|z| = r$. The function $\nu(r)$ is non-decreasing, integer valued, unbounded step function of r and has only left discontinuities. The maximum term and central index of $f(z)$ play a significant role in the study of the growth of an entire function. For an entire function $f(z)$ of exponential type, the growth number γ and δ, are defined by

$$(1.3.11) \qquad \lim_{r\to\infty} \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \right\} \frac{\nu(r)}{r} = \begin{matrix} \gamma \\ \delta \end{matrix}$$

Shah [77] proved that

(1.3.12) $$\delta \leq \left(\frac{\gamma}{e}\right) e^{\delta/\gamma} \leq \tau \leq \gamma$$

If $\{ |a_n/a_{n+1}| \}_{n=1}^{\infty}$ is a non-decreasing sequence of n then [39]

(1.3.13) $$\lim_{n \to \infty} \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \right\} n \left| \frac{a_n}{a_{n+1}} \right| = \begin{matrix} \gamma \\ \delta \end{matrix}$$

Let

(1.3.14) $$\Psi(z) = \sum_{n=o}^{\infty} e_n z^n$$

where $e_o = 1$, $e_n = (d_n \cdots d_1)^{-1}$ and $\{d_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive numbers . We call $\Psi(z)$ a comparision function , if e_{n+1}/e_n decreases to o as $n \to \infty$. A comparision function is necessarily entire . When $\Psi(z)$ is a comparision function , in analogy with the exponential type , Nachbin [59] defined Ψ-type of an entire function . Thus , an entire function f is said to be of finite Ψ-type , if there exists a constant B such that

(1.3.15) $$|f(re^{i\theta})| \leq M_o \, \Psi(Br)$$

for sufficiently large values of r and where M_o is a constant. The greatest lower bound of all such numbers B is called Ψ-type of f and is denoted by $\tau_\Psi(f)$. The characterization of the Ψ-type of an entire function $f(z) = \sum_{n=o}^{\infty} a_n z^n$ in terms of its coefficients a_n is given by

(1.3.16) $$\tau_\Psi(f) = \lim_{n \to \infty} \sup \left| \frac{a_n}{e_n} \right|^{\frac{1}{n}}$$

It is readily seen that if $d_n \equiv n$, then $\Psi(z) = e^z$ and (1.3.16) gives the coefficient characterization (1.3.10) for exponential type of an entire function.

1.4 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in $|z| < R$, $0 < R \leq \infty$. Let $\{d_n\}_{n=1}^{\infty}$ denote a non-decreasing sequence of positive numbers and let D be the operator which transforms the function

$$(1.4.1) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

into

$$(1.4.2) \qquad Df(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1} .$$

For $k = 0, 1, 2, \ldots$, the k-th iterate of D is given by

$$(1.4.3) \qquad D^k f(z) = \sum_{n=k}^{\infty} d_n \cdots d_{n-k+1} \, a_n z^{n-k} = \sum_{n=k}^{\infty} \frac{e_{n-k}}{e_n} a_n z^{n-k}$$

where $e_0 = 1$ and $e_n = (d_n \cdots d_1)^{-1}$, $n = 1, 2, \ldots$. We denote $D^0 f \equiv f$ and observe that $D^1 f = Df$. If $d_n \equiv n$, the operator D corresponds to the ordinary derivative , whereas , if $d_n \equiv 1$, D reduces to the shift operator L which transforms the function $f(z)$, given by (1.4.1), into $Lf(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$. The function f , Lf , $L^2 f$, ... are the normalized reminders of f . The operator D , called the Gelfond-Leontev derivative was introduced and studied by Gelfond and Leontev [26] in connection with the generalization of Fourier series.

Kazmin [45] investigated to some extent the operator D in relation to interpolation problems concerning analytic and, in particular, entire functions.

Buckholtz and Frank[15,16] considered the Gelfond-Leontev derivatives D to develop a general theory for the polynomial expansions of entire functions of exponential type and to study the univalence of successive ordinary derivatives of entire functions. Recently, Juneja and Shah [41] initiated the study of univalence of successive Gelfond-Leontev derivatives $D^k f$ for a function f analytic in $|z| < R$, $0 < R \leq \infty$ However, not much work has been done in this direction.

Some elementary properties of the operator D^k, $k = 1, 2, \ldots$ are as follows : For f and g analytic in $|z| < R$, $0 \leq R < \infty$ we have, for $k = 1, 2, \ldots,$

1. $D^k(f+g) = D^k f + D^k g$,
2. $D^k(\lambda f) = \lambda D^k f$, for all complex numbers λ.
3. $D^k(D^m f) = D^{k+m} f$, $m = 0, 1, 2, \ldots$
4. $D^k(f(\lambda z)) = \lambda^k D^k f(z)$ for all complex number λ such that $|\lambda| \leq 1$.

The function $\Psi(z)$, defined by (1.3.14) bears the same relationship to the operator D that the exponential function bears to the ordinary differentiation. That is, $\Psi(0) = 1$ and $D\Psi(z) = \Psi(z)$.

We now give some integral representations for entire functions of finite Ψ-type.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of finite Ψ-type (c.f.section 1.3) and

$$(1.4.4) \qquad \hat{f}(w) = \sum_{n=0}^{\infty} \frac{a_n}{e_n w^{n+1}}$$

be the Laplace transform of f. Then, $\hat{f}$ is analytic outside a disc with centre at origin and radius greater than Ψ-type of f. Using transform (1.4.4), the following integral representation of f is obtained [8] :

Let $\tilde{\Omega}(f)$ denote the union of the set of all singular points of f and the set of all points exterior to the domain of analyticity of f. If $\Psi(z)$, defined by (1.3.14), is a comparision function and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of finite Ψ-type, then

$$(1.4.5) \qquad f(z) = \frac{1}{2\pi i} \int_{\Gamma} \Psi(zw)\hat{f}(w)\,dw$$

where Γ is any contour enclosing the closed set $\tilde{\Omega}(f)$ and $\hat{f}$ is the Laplace transform of f given by (1.4.4).

If f is the entire function of exponential type τ_1, then $\tilde{\Omega}(f)$ lies in the disc $|w| \leq \tau_1$ and Γ may be taken as the circle $|w| = r > \tau_1$. When $\Psi(z) = e^z$, (1.4.5) reduces to the familiar representation for entire functions of exponential type ; i.e. if $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is entire function of order 1 and finite exponential type, then

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} e^{zw} \hat{f}(w)\, dw$$

where, $\hat{f}(w) = \sum_{n=o}^{\infty} \frac{a_n}{w^{n+1}}$ and Γ is the closed contour enclosing the set $\tilde{Q}(f)$.

The function $\Psi(z)$ can also be used to introduce a very useful class of polynomials $\mathbb{P}_n(z)$, called generalized Appell polynomials, which are defined by the formal relation [7].

(1.4.6) $$A(w)\Psi(zg(w)) = \sum_{n=o}^{\infty} \mathbb{P}_n(z) w^n$$

where,

(1.4.7) $$A(w) = \sum_{n=o}^{\infty} A_n w^n,\ A_o \neq o \text{ and } g(w) = \sum_{n=o}^{\infty} g_n w^n,\ g_o \neq o.$$

It is easy to check that $\mathbb{P}_n(z)$ is a polynomial of degree n. When $g(w) \equiv w$, $\Psi(z) = e^z$, the relation (1.4.6) gives Appell polynomials . In (1.4.6) the choice $g(w) = w$ gives the class of Brenke polynomials [11] and the particular case $\Psi(z) = e^z$ gives Sheffer polynomials [91]. Boas and Buck [8] characterized the generalized Appell polynomials as

(1.4.8) $$\mathbb{P}_n(z) = \sum_{j=o}^{n} z^j e_j \sum g_{k_o} g_{k_1} \cdots g_{k_j}$$

where the summation extends over all sets of (j+1) non-negative integers $\{k\}$ such that $k_o+k_1+\ldots+k_j = n$.

If $\Psi(z) = e^z$ and $g(w) = w$, (1.4.8) reduces to the well-known formula

$$\tilde{p}_n(z) = \sum_{j=o}^{n} \frac{a_{n-j}}{j!} z^j$$

for Appell polynomials characterised by the recursion formula

$$\tilde{p}_n'(z) = \tilde{p}_{n-1}(z).$$

Certain entire functions of finite Ψ-type can be represented in terms of a series of generalized Appell polynomials $\tilde{p}_n(z)$: Suppose $\Psi(z)$, defined by (1.3.14), is a comparision function. Also, suppose that $A(w)$ and $g(w)$ defined by (1.4.7) are analytic at o. Let $\tilde{\Omega}_w$ be a region in the w-plane in which $A(w)$ is analytic and $g(w)$ is analytic and univalent. Let τ_o be the distance from the origin to the nearest point of the boundary of $\tilde{\Omega}_w$. Then, the series in (1.4.6) is convergent for all w in the open disc $\Delta_w = \{w : |w| < \tau_o\}$. Let $\zeta = g(w)$ maps $\tilde{\Omega}_w$ onto a set $\tilde{\Omega}_\zeta$ in the ζ-plane, and denote the image of Δ_w by $\Delta_\zeta \subset \tilde{\Omega}_\zeta$. Let the inverse of g be $w = W(\zeta)$. Set $A(W(\zeta)) = B(\zeta)$.

Let K_o be any compact subset of Δ_ζ and denote $R_\Psi[K_o]$ the class of entire functions of finite Ψ-type with $\tilde{\Omega}(f) \subset K_o$. Let Γ be a simple closed contour lying in Δ_ζ and enclosing K_o but not passing through any zero of $B(\zeta)$. Then [8], any $f \in R_\Psi[K_o]$ has the convergent expansion

$$f(z) = \sum_{n=0}^{\infty} \pounds_n(f) \mathbb{P}_n(z) \tag{1.4.9}$$

where

$$\pounds_n(f) = \frac{1}{2\pi i} \int_\Gamma \frac{(W(\zeta))^n}{B(\zeta)} \hat{f}(\zeta)\, d\zeta$$

and $\hat{f}$ is the Laplace transform of f defined by means of $\Psi(z\zeta)$ as in (1.4.4).

If $g(w) = w$, then (1.4.6) becomes

(1.4.10) $$A(w)\Psi(zw) = \sum_{n=o}^{\infty} \mathbb{P}_n(z)w^n, \quad A(o) \neq o$$

and we have, if $\mathbb{P}_n(z)$ are the Appell polynomials defined by (1.4.10), $A(w)$ is analytic in the region $\tilde{\Omega}_o$ and Δ_o is the largest circular disc, with centre at o in $\tilde{\Omega}_o$ having radius ρ_o, then every entire function of Ψ-type less than ρ_o has the representation

(1.4.11) $$f(z) = \sum_{n=o}^{\infty} \mathcal{L}_n(f)\mathbb{P}_n(z)$$

where

$$\mathcal{L}_n(f) = \frac{1}{2\pi i} \int_{\Gamma} \frac{w^n}{A(w)} \hat{f}(w)\,dw$$

$\hat{f}(w)$ being the Laplace transform of $f(z)$, Γ is a circumference $|w| = r$ with $r < \rho_o$ on which $A(w) \neq o$.

1.5 The theory of univalent functions and the theory of entire functions have separately developed to great heights during last several decades. However, the work done to find underlying interconnections between the two kind of functions of seemingly different nature is of recent origin. The present and next two sections deal with some definitions and results in this direction.

Let E denote the class of functions $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ in S such that for each $k \geq 1$, $f^{(k)}$ is univalent in Δ. Let

$$\alpha_o = \sup \{ |a_2| : f \in E \}$$

Shah and Trimble [81] showed that if $f \in E$ then ,

$\pi/2 \leq \alpha_o < 1.7208$ and

$$(1.5.1) \qquad |a_n| \leq \frac{(2\alpha_o)^{n-1}}{n!}, \quad n = 2, 3, \ldots$$

$$(1.5.2) \qquad |f(z)| \leq \frac{(\exp(2\alpha_o|z|-1)}{\alpha_o}$$

$$(1.5.3) \qquad |f^{(n)}(z)| \leq (2\alpha_o)^{n-1}\exp(2\alpha_o|z|); \quad n = 1, 2, \ldots$$

Lachance [48] further proved that $\alpha_o < 1.5$

It follows from (1.5.1) and (1.3.10) that if $f \in E$, then f is entire and the exponential type $\tau_1 \equiv \tau_1(f) = \lim_r \sup(\log M(r)/r)$ of f satisfies $\tau_1 \leq 2\alpha_o$. Let

$$\tilde{\tau} = \sup\{\tau_1(f) : f \in E\}$$

Since the function $(e^{\pi z}-1)/\pi$ in E has exponential type π , we have

$$\pi \leq \tilde{\tau} < 2\alpha_o$$

Further , it follows from (1.5.2) that the class E is a normal family [41] .

Recently , Juneja and Shah [41] extended the results (1.5.1) to (1.5.3) for Gelfond-Leontev derivatives. To discuss their results , define the Ψ-type of a function $f(z) = \sum_{n=o}^{\infty} a_n z^n$ analytic in a neighborhood of origin , by

$$(1.5.4) \qquad \tau_\Psi(f) = \lim_{n \to \infty} \sup \left|\frac{a_n}{e_n}\right|^{\frac{1}{n}} = \lim_{n \to \infty} \sup |D^n f(o)|^{\frac{1}{n}}$$

Let $r_o(\Psi)$ be the radius of convergence of $\Psi(z)$, defined by (1.3.14). From the monotonicity of $\{d_n\}_{n=1}^{\infty}$, we have

$$r_o(\Psi) = \lim_{n\to\infty} d_n = \sup_{n\geq 1} \{d_n\}$$

If $r_o(\Psi) < \infty$ and f has radius of convergence R , $o < R < \infty$, then from (1.5.4) , we obtain

(1.5.5) $$\tau_\Psi(f) = \frac{r_o(\Psi)}{R}$$

Clearly , from (1.5.4) , f and all its Gelfond-Leontev derivatives $D^k f$, $k = 1, 2, \ldots$ have the same Ψ-type . From (1.5.5) it follows that if $\tau_\Psi(f) < r_o(\Psi)$, then each of f , Df , ... is analytic in the unit disc Δ . If f is an entire function , to avoid trivial case , $\Psi(z)$ is always taken to be a comparison function (c.f. Section 1.3) and so , the Ψ-type of an entire function defined by (1.5.4) becomes the same as Nachbin's Ψ-type defined in section 1.3 . It may be further seen that if $\Psi(z)$ is an entire function and $\tau_\Psi(f)$ is finite , then (1.5.5) gives that f is entire.

Let E(D) be the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ such that f and all its Gelfond-Leontev derivatives $D^k f$, $k = 1, 2, \ldots$ are analytic and univalent in Δ . If $f \in E(D)$, then [41]

(1.5.6) $$|a_k| \leq e_k (2d_2)^{k-1} d_1 , \quad k = 2, 3, \ldots$$

(1.5.7) $$|f(z)| \leq \frac{d_1}{2d_2}(\Psi(2d_2|z|)-1), \quad |z| < \frac{r_o(\Psi)}{2d_2}$$

and

(1.5.8) $$|D^k f(z)| \leq d_1 (2d_2)^{k-1}(\Psi(2d_2|z|)-1), \quad k = 1, 2, \ldots$$

From (1.5.6) and in view of (1.5.4), it is clear that functions in E(D) have finite Ψ-type not exceeding $2d_2$. Also from (1.5.7), E(D) is a normal family in $|z| < t < \frac{r_o(\Psi)}{2d_2}$ for all t satisfying $o < t < \frac{r_o(\Psi)}{2d_2}$.

Let f be analytic in a neighborhood of the origin. The radius of univalence $\tilde{\rho}_n$ of $f^{(n)}$, $n = 0,1,2,\ldots$ is defined to be the largest number with the property that $f^{(n)}$ is analytic and univalent in the open disc about the origin of radius $\tilde{\rho}_n$. Likewise, the radius of convexity $\tilde{\rho}_n(c)$ of $f^{(n)}$ is defined to be the largest number with the property that $f^{(n)}$ is analytic, univalent and convex in the open disc about the origin of radius $\tilde{\rho}_n(c)$.

If the derivatives $f^{(n)}$ are not necessarily univalent throughout Δ and have radius of univalence $\tilde{\rho}_n$, even then a relationship between the radius of convergence R of f and the growth of $\tilde{\rho}_n$ can be found. Similarly, relationship between R and the growth of $\tilde{\rho}_n(c)$ can also be found. Shah and Trimble [82,88], Buckholtz [13,14], Buckholtz and Frank [15,16] obtained several results of this nature. They also obtained relations between the growth of the sequence $\{\tilde{\rho}_n\}_{n=o}^{\infty}$ and the order and type (c.f. Section 1.3) of an entire function f. The following are some of the basic results in this direction which are relavant to the present study:

Let f, defined by,

(1.5.9) $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

have radius of convergence R, $0 < R \leq \infty$. If the function f, given by (1.5.9), is not a polynomial and if

$$\underline{R} = \liminf_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

then [82], $\underline{R} \leq R$ and

(1.5.10) $$\liminf_{n \to \infty} n\tilde{\rho}_n \leq 4\underline{R}$$

(1.5.11) $$\limsup_{n \to \infty} n\tilde{\rho}_n \geq R \log 2.$$

The inequality (1.5.11) has been further sharpened as

(1.5.12) $$\limsup_{n \to \infty} n\tilde{\rho}_n \geq RW$$

where W is the Whittaker constant [13].

If $\lim_{n \to \infty} n\tilde{\rho}_n = \infty$, then (1.5.10) gives that f is a transcendental entire function. So that if $\tilde{\rho}_n$ converges to zero slowly enough then f is an entire function. However, the converse is false [82]. Similarly, the inequality (1.5.11) gives that if f is a transcendental entire function, then $\limsup_{n \to \infty} n\tilde{\rho}_n = \infty$. The converse of this is also not true [82]. There are functions for which $\underline{R} = 0$, $R = 1$ and so in view of (1.5.10) and (1.5.11), $\lim_{n \to \infty} n\tilde{\rho}_n$ need not always exist.

For a function f analytic in a neighborhood of the origin, the radius of univalence ρ_n of $D^n f$, $n = 0, 1, 2, \ldots$ is defined

to be the largest number with the property that $D^n f$ is analytic and univalent in the open disc about the origin of radius ρ_n. Similarly, let $\rho_n(c)$ be the radius of convexity of $D^n f$. Recently, Juneja and Shah [41] have extended the inequality (1.5.10) in the context of Gelfond-Leontev derivatives, i.e., if f, given by (1.5.9), have radius of convergence R, $0 < R \leq \infty$, then

$$\text{(1.5.13)} \qquad \liminf_{n\to\infty} d_n \rho_n \leq \liminf_{n\to\infty} \Big[\prod_{k=N}^{n} d_k \rho_k\Big]^{\frac{1}{n}} \leq 2d_2 R$$

where N denotes the smallest non-negative integer such that for $n \geq N$, $\rho_n > 0$.

The inequalities (1.5.10) and (1.5.11) respectively give upper bound of $\liminf_{n\to\infty} n\tilde{\rho}_n$ and lower bound of $\limsup_{n\to\infty} n\tilde{\rho}_n$ in terms of the radius of convergence R of the function f defined by (1.5.9). The bounds of $\liminf_{n\to\infty} n\tilde{\rho}_n$ and $\limsup_{n\to\infty} n\tilde{\rho}_n$ in the other direction are obtained under an additional condition on the coefficients a_n of f in [82,88]. Thus, if f, given by (1.5.9), has radius of convergence R, $0 < R < \infty$, and $\lim_{n\to\infty} |a_n/a_{n+1}|$ exists, then

$$\text{(1.5.14)} \qquad \limsup_{n\to\infty} n\tilde{\rho}_n \leq 2\sqrt{3}\, R$$

Further, if $|a_n/a_{n+1}|$ is ultimately a non-decreasing sequence, then

$$\text{(1.5.15)} \qquad \liminf_{n\to\infty} n\tilde{\rho}_n \geq R \log 2$$

Similarly, if $\lim_{n\to\infty} |a_n/a_{n+1}|$ exists, then

$$\text{(1.5.16)} \qquad \limsup_{n\to\infty} n\tilde{\rho}_n(c) \leq \sqrt{2}\, R$$

Thus, if f, given by (1.5.9), is a transcendental entire function and $|a_n/a_{n+1}|$ is ultimately a non-decreasing sequence of n, then (1.5.15) implies that $\liminf_{n\to\infty} n\tilde{\rho}_n = \infty$. Since, for the function $f(z) = z(1-z)^{-1}$, $\tilde{\rho}_n = \sin(\pi/(n+1))$ and $\tilde{\rho}_n(c) = 1/(n+1)$, the constant $2\sqrt{3}$ on the right side of (1.5.14) can not be replaced by any number less than π and the constant $\sqrt{2}$ in (1.5.16) can not be replaced by any number less than 1.

In the context of Gelfond-Leontev derivatives, (1.5.14) is extended by Juneja and Shah [41]. Thus, if $f(z)=\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R, is of finite Ψ-type and $\{|a_n/a_{n+1}|\}$ is ultimately non-decreasing, then

$$\limsup_{n\to\infty} d_n\rho_n \leq d_2 R\sqrt{d_3/(d_3-d_2)} .$$

The growth of the sequence, $\{\tilde{\rho}_n\}_{n=0}^{\infty}$ is intimately related to the growth parameters like order Λ, lower order Λ_*, type τ and lower type τ_* (c.f. section 1.3) of an entire function f. Boas [5] showed that if f is a transcendental entire function of exponential type less than log2, then there is a subsequence, $\{\tilde{\rho}_{n_p}\}_{p=1}^{\infty}$, such that $\tilde{\rho}_{n_p} \geq 1$ for all p. He further showed that if the order of an entire function is less than one, or if, f is of order one and type zero, then $\limsup_{n\to\infty} \tilde{\rho}_n = 0$.

Polya [63] proved that

$$\liminf_{n\to\infty} \frac{\log \tilde{\rho}_n}{\log n} \leq \frac{1-\Lambda}{\Lambda} \leq \limsup_{n\to\infty} \frac{\log \tilde{\rho}_n}{\log n} .$$

Shah and Trimble [82] further improved these results and obtained several other results connecting the growth of the sequence $\{\tilde{\rho}_n\}_{n=0}^{\infty}$ with the growth parameters of an entire function f . Thus , if f , given by (1.5.9) , is a transcendental entire function , then

$$(1.5.17) \qquad \limsup_{n\to\infty} \frac{\log \tilde{\rho}_n}{\log n} \geq \frac{1-\Lambda_*}{\Lambda_*}$$

$$(1.5.18) \qquad \liminf_{n\to\infty} n^{\Lambda-1}\, \tilde{\rho}_n^{\Lambda} \leq \frac{4^{\Lambda}}{\Lambda\tau}$$

and

$$(1.5.19) \qquad \limsup_{n\to\infty} \tilde{\rho}_n \geq \frac{W}{\tau} .$$

where W is the Whittaker constant . The inequality (1.5.19) is due to Buckholtz [14].

Juneja and Shah [41] extended some of the results of Shah and Trimble [82] in the context of Gelfond-Leontev derivatives , thus connecting the growth of the sequence $\{\rho_n\}_{n=0}^{\infty}$ with the growth parameters of an entire function. In particular , they proved that if , f given by (1.5.9) , is a transcendental entire function of order Λ and type τ , then

$$(1.5.20) \qquad \liminf_{n\to\infty} \left[\frac{d_n \rho_{n-2}}{n^{1/\Lambda}}\right]^{\Lambda} \leq \liminf_{n\to\infty} \left[\prod_{k=N}^{n} \frac{d_k \rho_{k-2}}{k^{1/\Lambda}}\right]^{\Lambda} \leq \frac{(2d_2)^{\Lambda}}{\Lambda\tau}$$

where N denotes the non-negative integer such that for $n \geq N$, $\rho_n > 0$.

It follows from (1.5.20) that if $\liminf_{n\to\infty} d_n n^{-1/\Lambda} > 0$ and $\liminf_{n\to\infty} \rho_n = \infty$, then $\tau = 0$.

For an entire function f, an upper bound for $\tilde{\rho}_n$ and $\tilde{\rho}_n(c)$ in terms of the growth number γ and δ, defined by (1.3.9), is also obtained [88]. Thus, if f, given by (1.5.9), is entire,

$$(1.5.21) \qquad \limsup_{n\to\infty} \left| \frac{a_{n+1}\, a_{n-1}}{a_n^2} \right| \leq 1$$

and $|a_n/a_{n+1}|$ is non-decreasing sequence, then

$$(1.5.22) \qquad \lim_{n\to\infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \tilde{\rho}_n \leq \begin{matrix} 2\sqrt{3}/\delta \\ 2\sqrt{3}/\gamma \end{matrix}$$

and

$$(1.5.23) \qquad \lim_{n\to\infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \tilde{\rho}_n(c) \leq \begin{matrix} \sqrt{2}/\delta \\ \sqrt{2}/\gamma \end{matrix}$$

Since for the function $f(z) = e^z$, (1.5.21) is satisfied and $\tilde{\rho}_n = \pi$, $\tilde{\rho}_n(c) = 1$, the constant $2\sqrt{3}$ in (1.5.22) can not be replaced by any constant less than π and the constant $\sqrt{2}$ in (1.5.23) by a constant less than 1.

A function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is said to be defined by a gap series if $a_n \neq 0$, $\lambda_0 = 1$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers. Several results connecting the sequence $\{\tilde{\rho}_n\}_{n=0}^{\infty}$ with the growth parameters and growth numbers of an entire function f can be extended and improved if the function f has gaps. Read [68] showed that if f is an even transcendental entire function, then

the constant W in (1.5.9) can be replaced by $\log(2+\sqrt{3})$. A similar result for functions analytic in a disc was proved by Iyer [37] . Shah and Trimble [87, 88] extended this study and obtained a number of results in this direction with restrictive gaps.

Further, the conditions on the zeros of functions to be in the class E are studied by Shah and Trimble [84, 90] and Salmassi [72, 73] .

1.6 In this section we discuss some results for analytic functions with some of the derivatives univalent in the unit disc Δ .

From (1.5.10), it follows that if $\tilde{\rho}_n \equiv 1$, then f is an entire function i.e., $R = \infty$. However, if infinitely many $\tilde{\rho}_n$'s are zero, then (1.5.10) gives no non-trivial information about the radius of convergence R of f, given by (1.5.9) . For instance, the function $2\sin(\pi z/2)/\pi$ is entire while (1.5.10) only gives that its radius of convergence is greater than equal to zero . This shortcoming is partially overcome by considering the derivatives $f^{(n_p)}$ of f to be univalent in Δ for a strictly increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers.

Let $E(n_p, R)$ denote the class of functions $f(z) = \sum_{n=o}^{\infty} a_n z^n$ analytic in Δ and having radius of convergence R, $o < R \leq \infty$,

such that $f^{(n_p)}$, $p = 0,1,2,\ldots$ is univalent in Δ for a strictly increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers. Shah and Trimble [83] proved that if $f \in E(n_p; R)$, then the following relation between the radius of convergence R and the sequence $\{n_p\}^{\infty}$ hold :

$$(1.6.1) \qquad \liminf_{p\to\infty} (n_1 n_2 \cdots n_p)^{1/n_p} \leq R \limsup_{p\to\infty} 4^{p/n_p} \leq 4R .$$

From (1.6.1), it is clear that if

$$(1.6.2) \qquad (n_{p+1}-n_p) = o(\log n_p)$$

then f is entire. Here and in the sequel $q_1(x) = o(q_2(x))$ is used to mean that the functions q_1 and q_2 are defined for $x > x_0$, $q_2(x) \neq 0$ and $q_1(x)/q_2(x) \to 0$ as $x \to \infty$.

Further, if there is a integer $M > 1$ such that $\liminf_{p\to\infty}(n_{p+1}-n_p) \geq M$, then

$$(1.6.3) \qquad \liminf_{p\to\infty} [(n_1 n_2 \cdots n_p)^{1/n_p}]^{M-1} \leq R$$

It follows from (1.6.3) that if $\lim_{p\to\infty} (n_{p+1}-n_p) = \infty$ and $\liminf_{p\to\infty} (n_1 n_2 \cdots n_p)^{1/n_p} > 1$, then f is an entire function

Suppose $f \in \tilde{E}(n_p, R)$ and let

$$\alpha_0 = \liminf_{p\to\infty} \frac{n_p}{n_{p+1}}.$$

A relation between R and α_0 is obtained in [85]. Thus, it is proved that

(1.6.4) If $\alpha_0 = 1$, then $R = \infty$, i.e. f is entire

(1.6.5) If $0 < \alpha_o < 1$, then $R \geq \frac{\alpha_o^{\alpha_o/(\alpha_o-1)}}{(1-\alpha_o)}$

(1.6.6) If $\alpha_o = 1$, then $R \geq 1$.

The inequality (1.6.5) is Sharp [86] . We note that the condition $\alpha_o = 1$ is weaker than (1.6.2). Further, it is shown that if $n_{p+2}-2n_{p+1}+n_p = o(n_p)$, then f is an entire function.

A function $s(x)$ continuous on $[1,\infty)$, is said to be slowly oscillating if for every positive number $c > o$,

$$\lim_{x \to \infty} \frac{s(cx)}{s(x)} = 1.$$

Shah and Trimble [83,86] have given some other sufficient conditions on the sequence $\{n_p\}_{p=1}^{\infty}$, also, restricting the growth of the sequence $\{n_p-n_{p-1}\}_{p=2}^{\infty}$ by slowly oscillating functions [35, 44] which forces the functions f to be entire of order not exceeding a finite number depending upon the slowly oscillating functions .

In [86], they obtained an upper bound on the order of an entire function f in terms of the exponents $\{n_p\}$ for which $f^{(n_p)}$ is univalent in Δ. Thus, if $f \in E(n_p,\infty)$ and $\log n_p \sim \log n_{p+1}$, i.e., $\lim_{p \to \infty} (\log n_p/\log n_{p+1}) = 1$, then

(1.6.7) $$\Lambda \leq \frac{1}{1-\limsup_{p \to \infty} \dfrac{\log(n_p-n_{p-1})}{\log n_p}}$$

It follows from (1.6.7) that if (1.6.2) holds, then $\Lambda \leq 1$. Also, if the growth condition on $\{n_p\}_{p=1}^{\infty}$ is taken as

$n_p \sim n_{p+1}$, then (1.6.4) gives that f is entire .

Let

$$(1.6.8) \qquad \Lambda^* = \frac{1}{1-\limsup_{p\to\infty} \frac{\log(n_p-n_{p-1})}{\log n_p}}$$

Obviously , $\Lambda^* \geq 1$. So , if $0 \leq \Lambda \leq 1$, then (1.6.7) gives no information about the relationship between the sequence $\{n_p\}_{p=1}^{\infty}$ and the order Λ of the entire function f . Infact , Shah and Trimble [86] showed that no such relation exists by showing that, if $o \leq \Lambda \leq 1$, then there exists an entire function f of order Λ such that $f^{(n)}$ is univalent in Δ , if and only if , $n = n_p$ for some p . It follows therefore that the inequality (1.6.7) is sharp for $\Lambda = 1$. It is also shown that if Λ , $1 < \Lambda < \infty$, is given then there exists a sequence $\{n_p\}_{p=1}^{\infty}$ and an entire function f of order Λ such that $f^{(n_p)}$ is univalent in Δ and equality holds in (1.6.7). It is natural to enquire therefore that given Λ, $1 < \Lambda < \infty$, and a sequence $\{n_p\}_{p=1}^{\infty}$ of strictly increasing sequence of positive integers , does there exists an entire function such that $f^{(n_p)}$ is univalent in Δ and equality holds in (1.6.7). The answer to this query is also given in affirmative in [86].

If Λ^* defined by (1.6.8) is finite , then f is an entire function of order no greater than Λ . If $\Lambda^* = \infty$, then f need not be entire and if it is entire , it may be of

any oder. This is in analogy to the discussions in Section 1.3 for the coefficient characterization of the order of an entire function.

The following result of Shah and Trimble [86] gives an upper bound on the exponential type τ_1 of an entire function in terms of the exponents $\{n_p\}_{p=1}^{\infty}$ for which $f^{(np)}$ is univalent in Δ. Thus, if $f \in \tilde{E}(n_p, 1)$ and

(1.6.9) $$\lim_{p \to \infty} \sup (n_{p+1} - n_p) = \mu < \infty$$

then f is an entire function of exponential type $\tau_1 (\equiv \lim_r \sup (\log M(r,f)/r))$ satisfying

(1.6.10) $$\tau_1 \leq e(\frac{7\pi}{3})^{1/2} e^{1/24} (\mu+1)^{7/2}.$$

This result has been further improved and extended by Shah [80]. Let f be analytic in Δ and $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $f^{(n_p)}$ is univalent in Δ. Set,

$$\tilde{\eta}_p = \sum_{j=1}^{p} \log \{(n_j - n_{j-1})!\};\ \tilde{\xi}_p = \log n_p - \frac{\tilde{\eta}_p}{n_p};\ p=1,2,\ldots$$

$$\lim_{p\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\tilde{\eta}_p}{n_p} = \tilde{\eta}\ ;\ \lim_{p\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{p}{n_p} = \begin{matrix}\tilde{\theta}_1\\ \tilde{\theta}_2\end{matrix}$$

Then, it is proved [80] that

(1.6.11) If $\lim_{p \to \infty} \tilde{\xi}_p = \infty$, then f is entire.

If $\tilde{\eta} < \infty$, $\tilde{\theta}_1 > \tilde{\theta}_2 > 0$ and f is an entire function of exponential type τ_1 satisfying

(1.6.12) $$\tau_1 \leq \{\log c_1 + \tilde{\eta}(2 - \frac{\tilde{\theta}_1}{\tilde{\theta}_2}) + 2\tilde{\theta}_1 \log(1 + \frac{1}{\tilde{\theta}_1})\}$$

where $1 \leq c_1 \leq 1.0691$, is a constant.

(1.6.13) If $\tilde{\eta} = \infty$ and $\lim_{p\to\infty} \tilde{\xi}_p = \infty$, then f is entire but not necessarity of exponential type. If $\liminf_{p\to\infty} \tilde{\xi}_p < \infty$, then f may not be an entire function.

If the sequence $\{n_p\}_{p=1}^{\infty}$ satisfies (1.6.9), then the following results better than (1.6.10) and (1.6.12) are obtained

(1.6.14) If $\mu = 1$, then $\tau_1 \leq 2\sqrt{3}$

(1.6.15) $$\tau_1 \leq \exp\{\tilde{\theta}_1 \log c_1 + \tilde{\eta} + 2\tilde{\theta}_1 \log(1 + \frac{1}{\tilde{\theta}_1})$$

(1.6.16) $$\leq c_1^{\tilde{\theta}_1}\{(\mu+1)!(\mu+1)\}^{1/\mu}.$$

Further,

(1.6.17) If $(n_{p+1} - n_p) \to \mu$ as $p \to \infty$, then
$$\tau_1 \leq \{c_1(\mu+1)!(\mu+1)\}^{1/\mu}.$$

1.7 Some of the subclasses of E(c.f. Section 1.5) having many interesting properties are studied by Buckholtz and Shah [17,18]. We give in this section basic definitions and results concerning these subclasses.

Let $\{b_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that

(1.7.1) $$b_1 = 1, \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{j=n}^{n+k-1} b_{j+2} \leq 1, \quad n = 0,1,2,\ldots$$

Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that

(1.7.2) $\quad a_0 = 0 , \; a_1 = 1$ and $\left|\frac{a_{n+1}}{a_n}\right| \leq \frac{b_{n+1}}{n+1}$, $n = 1, 2, \ldots$

Let $\mathbb{E}$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for which there is a sequence of positive numbers , $\{b_j\}_{j=1}^{\infty}$, satisfying (1.7.1) and (1.7.2).

If $f \in \mathbb{E}$, then Buckholtz and Shah [18] proved that f is starlike univalent in Δ and , for each $n \geq 1$, $f^{(n)}$ is univalent in Δ . Since , the function $(e^{\pi z}-1)/\pi$ is in E but not in $\mathbb{E}$, the class $\mathbb{E}$ is properly contained in E. Also, it is proved in [18] that

(1.7.3) $$\alpha^* = \{\sup |a_2| : f \in \mathbb{E}\} = \frac{1}{2}$$

(1.7.4) $$\tau^* = \sup \{\tau_1(f) = \limsup_{r \to \infty} \frac{\log M(r)}{r} ; f \in \mathbb{E}\} = \log 2$$

(1.7.5) $$|f(z)| \leq \frac{2}{e\log 2} \; \frac{2^{|z|}-1}{\log 2}$$

and

(1.7.6) $$|f'(z)| \leq \frac{2}{e\log 2} \; 2^{|z|}$$

for all z . Further , if equality holds in (1.7.1) for some positive sequence $\{b_j\}_{j=1}^{\infty}$ such that $b_1 = 1$, then

$$b_j = \log 2 , \; j \geq 2 .$$

Recently , Juneja and Shah [41] extended the above class to the class $\mathbb{E}(D)$ consisting of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ provided there is a sequence $\{b_j\}_{j=1}^{\infty}$ of positive numbers , satisfying

$$(1.7.7) \qquad b_1 = 1, \sum_{k=1}^{\infty} \frac{(k+1)}{d_{k+1}\cdots d_2} \prod_{j=n}^{n+k-1} b_{j+2} \leq 1 \ , \ n = 0,1,2,\ldots$$

and

$$(1.7.8) \qquad \left|\frac{a_{n+1}}{a_n}\right| \leq \frac{b_{n+1}}{d_{n+1}} \ , \ n = 1,2,\ldots$$

They proved that if $f \in \mathbb{E}(D)$, then f is starlike univalent in Δ and for each $k \geq 1$, $D^k f$ is univalent in Δ . The class $\mathbb{E}(D)$ is properly contained in the class $E(D)$ (c.f. Section 1.5). The class $\mathbb{E}(D)$ may contain , in general , functions that are not entire. If $b_n/d_n = o(1)$, then $f \in \mathbb{E}(D)$ is necessarily entire . However , the converse is false.

Another subclass of E , that contains the class $\mathbb{E}$, is studied by Buckholtz and Shah [17] . Thus , we have

<u>Definition</u> 1.7.1 A function f analytic in the unit disc Δ and satisfying the in finite system of inequalities :

$$(1.7.9) \qquad \sum_{n=2}^{\infty} n \frac{|f^{(n+k)}(o)|}{n!} \leq |f^{(k+1)}(o)| \ , \ k = 0,1,2,\ldots$$

is called <u>absolute starlike</u> .

For an absolute starlike function f and every nonnegative integer k the image of Δ under kth derivative $f^{(k)}$ is starlike with respect to the point $f^{(k)}(o)$. Absolute starlike functions are necessarily entire and of finite exponential type [81]. Let A_s denote the class of absolute starlike functions f for which $f(o) = f'(o)-1 = o$ and B denote the subclass of A_s

consisting of those functions f for which $f^{(k)}(o)$ is real and non-negative for each $k = 2, 3, \dots$.

<u>Definition</u> 1.7.2 A function f analytic in the unit disc Δ and satisfying the infinite system of inequalities

$$(1.7.10) \qquad \sum_{n=2}^{\infty} n^2 \frac{|f^{(n+k)}(o)|}{n!} \leq |f^{(k+1)}(o)| , \quad k = o, 1, 2, \dots$$

is called <u>absolute convex</u> .

For an absolute convex function f , the image of Δ under $f^{(k)}$ is a convex set for every nonnegative integer k. Let A_c denote the class of absolute convex functions for which $f(o) = f'(o)-1 = o$ and B_c denote the subclass of A_c consisting of those functions f for which $f^{(k)}(o)$ is real and non-negative for each $k = 2, 3, \dots$.

To study certain extremal problems for the classes A_s and A_c , Buckholtz and Shah [17] defined the linear functionals $\tilde{U}_k$ and $\tilde{V}_k$ by

$$(1.7.11) \qquad \tilde{U}_k(f) = 2f^{(k)}(o)-f^{(k)}(1)$$

$$(1.7.12) \qquad \tilde{V}_k(f) = 2f^{(k)}(o)-f^{(k+1)}(1)-f^{(k+2)}(1)$$

for $k = o, 1, 2, \dots$.

It is easily seem that f is in B_s , if and only if ,

$$\tilde{U}_k(f) \geq o , \quad k = o, 1, 2, \dots$$

and f is in B_c , if and only if ,

$$\tilde{V}_k(f) \geq o , \quad k = o, 1, 2, \dots$$

The representation of entire functions of exponential type in terms of a series of Appell polynomials play an important role in the investigation of certain extremal problems for the class A_s and A_c. The Appell polynomial sequences $\{\tilde{p}_n(z)\}_{n=0}^{\infty}$ and $\{\tilde{q}_n(z)\}_{n=0}^{\infty}$ are defined by the relation

(1.7.13) $$\sum_{n=0}^{\infty} \tilde{p}_n(z) w^n = e^{zw}(2-e^w)^{-1}$$

and

(1.7.14) $$\sum_{n=0}^{\infty} \tilde{q}_n(z) w^n = e^{zw}(2-(1+w)e^w)^{-1}$$

Also, let

(1.7.15) $$\sum_{n=0}^{\infty} \tilde{u}_n w^n = (2-e^w)^{-1}$$

and

(1.7.16) $$\sum_{n=0}^{\infty} \tilde{v}_n w^n = (2-(1+w)e^w)^{-1}.$$

Then, by (1.7.15), we get the following representation of $\tilde{p}_n(z)$ in terms of $\{\tilde{u}_k\}_{k=0}^{\infty}$

$$\tilde{p}_n(o) = \tilde{u}_n \; ; \; \tilde{p}_n(z) = \sum_{k=0}^{n} \tilde{u}_{n-k} \frac{z^k}{k!}$$

Similarly, (1.7.16) gives

$$\tilde{q}_n(o) = \tilde{v}_n \; ; \; \tilde{q}_n(z) = \sum_{k=0}^{n} \tilde{v}_{n-k} \frac{z^k}{k!}$$

Now, by normalizing the polynomials $\tilde{p}_n(z)$ and $\tilde{q}_n(z)$, define the polynomials $\tilde{P}_n(z)$ and $\tilde{Q}_n(z)$ as

$$\tilde{P}_n(z) = \frac{\tilde{p}_n(z)-\tilde{p}_n(o)}{\tilde{p}_n'(o)},$$

$$\tilde{Q}_n(z) = \frac{\tilde{q}_n(z)-\tilde{q}_n(o)}{\tilde{q}'_n(o)} .$$

Buckholtz and Shah [17] proved that

(1.7.19) $$\lim_{n\to\infty} \tilde{P}_n(z) = \tilde{F}_s(z)$$

(1.7.20) $$\lim_{n\to\infty} \tilde{Q}_n(z) = \tilde{F}_c(z)$$

uniformly on compact sets, where $\tilde{F}_s(z) = (2^z-1)/\log 2 \in B_s$, $\tilde{F}_c(z) = (e^{\tilde{\alpha}z}-1)/\tilde{\alpha} \in B_c$ and $\tilde{\alpha}$ is the root of the equation $(1+w)e^w = 2$.

Using (1.4.11), the polynomial representation formulae for entire functions of exponential type less than log2 or $\tilde{\alpha}$. is found in [17]. Thus, if f is an entire function of exponential type less than log2 and f(o) = o, then

(1.7.21) $$f(z) = \sum_{n=o}^{\infty} \tilde{U}_{n+1}(f)\, \tilde{u}_n \tilde{P}_{n+1}(z).$$

uniformly on compact sets. The function $f(z) = \tilde{F}_s(z) \in B_s$ shows that the representation (1.7.21) is not valid for every function in B_s.

Likewise, if f is an entire function of exponential type less than $\tilde{\alpha}$ with f(o) = o, then

(1.7.22) $$f(z) = \sum_{n=o}^{\infty} \tilde{V}_{n+1}(f)\tilde{v}_n\, \tilde{Q}_{n+1}(z)$$

The function $f(z) = \tilde{F}_c(z) \in B_c$ shows that the representation (1.7.22) is not valid for every function in B_c.

Using the representation formula (1.7.21) and (1.7.22), the sharp coefficient bounds for functions in A_s and A_c can be obtained [17]. Thus, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in A_s, then

$$|a_n| \le \frac{-1}{n!\tilde{u}_{n-1}}, \quad n = 2, 3, \ldots \tag{1.7.23}$$

and the inequality in (1.7.23) is sharp. If f in A_c, then

$$|a_n| \le \frac{1}{n!\,\tilde{v}_{n-1}} \tag{1.7.24}$$

and the inequality in (1.7.24) is sharp.

The following simplicial representation formula hold for functions in the class B_s or B_c [17]. If $f \in B_s$, then

$$f(z) = \tilde{S}(f)\tilde{F}_s(z) + \sum_{k=0}^{\infty} \tilde{U}_{k+1}(f)\, \tilde{u}_k\, \tilde{P}_{k+1}(z) \tag{1.7.25}$$

where

$$\tilde{S}(f) = 1 - \sum_{k=0}^{\infty} \tilde{U}_{k+1}(f) \ge 0$$

Similarly, if $f \in B_c$, then

$$f(z) = \tilde{C}(f)\tilde{F}_c(z) + \sum_{k=0}^{\infty} \tilde{V}_{k+1}(f)\tilde{v}_k\, \tilde{Q}_{k+1}(z) \tag{1.7.26}$$

where,

$$\tilde{C}(f) = 1 - \sum_{k=0}^{\infty} \tilde{V}_{k+1}(f) \ge 0$$

The expansions are valid for all z and convergence of the infinite series is uniform on every compact set.

The classes A_s and A_c are related by the following [17]. If $f \in A_s$, then $2f(z/2) \in A_c$. Conversely, if $f \in A_c$, then

the function $f(\lambda_o z)/\lambda_o \in A_s$ where λ_o is the positive root of the equation $2 - \tilde{Q}_9'(z) = o$. The inequality (1.7.23) and (1.7.24) can be used to show that every absolute starlike function f satisfies

$$f^{(n)}(o) = O(\log 2) \tag{1.7.27}$$

and every absolute convex function satisfies

$$f^{(n)}(o) = O(\tilde{\alpha}) \tag{1.7.28}$$

where $\tilde{\alpha}$ is the root of the equation $2-(1+w)e^w = o$. Here and in the sequel $q_1(x) = O\ (q_2(x))$ is used to mean that the functions q_1 and q_2 are defined for $x > x_o$, $q_2(x) \neq o$ and $q_1(x)/q_2(x)$ is bounded as $x \to \infty$.

Finally, if is known [17] that entire functions satisfying (1.7.27) can be represented as the sum of two absolute starlike function and every entire function satisfying (1.7.28) can be represented as the sum of two absolute convex functions.

1.8 The theory of univalent functions has a vast literature spanning over several thousand research papers. The theory of entire functionsis even more vast. Therefore, to do justice to our presentation in this brief introduction, we have to confine ourselves in previous sections to the discussions of only those concepts and results from these two enormous fields that are relevant to our present study.

The discussions in Sections 1.5 to 1.7 reveal that analytic functions with univalent ordinary derivatives have been deeply studied during last two decades or so and that only a few authors studied analytic functions with univalent Gelfond-Leontev derivatives. Though , the later study is natural , non-trivial and much more general than the earlier study , very little progress has been made in this area of investigation. Thus , there is a clear need to study in detail the consequences of univalence of Gelfond-Leontev derivatives of analytic functions and investigate related function classes arising in the processes . Our present work is an endeavour in this direction.

The thesis is divided into six chapters , Chapter I being the introduction.

In Chapter II , we study classes of functions of the form $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ that are analytic and univalent in the unit disc Δ and have some or all the Gelfond-Leontev derivatives $D^k f$, $k = 1, 2, \ldots$, analytic and univalent (or analytic and convex) in Δ . We find sufficient conditions on the coefficients $|a_n|$ for functions f to be in these subclasses. For functions , specially different type of polynomials belonging to some of these subclasses, necessary conditions on the coefficients are also obtained. Further , the sharp upper bound for the second coefficient of functions whose all the Gelfond-Leontev derivatives are analytic and

univalent (or analytic and convex) in Δ is found . Besides giving several new results , our investigations in this chapter generalize the recent work of Silverman [94].

Chapter III deals with the study of functions f , analytic in $|z| < R$, $0 < R \leq \infty$, with some Gelfond-Leontev derivatives analytic and univalent in Δ . We find relations between the exponents n_p for which the Gelfond-Leontev derivatives $D^{n_p}f$ are analytic and univalent in Δ and the radius of convergence R of f . The relations between the exponents n_p and the type of an entire function are also obtained. Further , sufficient conditions on the exponents n_p that force the function f to be entire are found . Our results found in this chapter generalize some of the results of Shah and Trimble [83] and give a number of new results.

Chapter IV deals with the growth of entire functions f when some of the Gelfond-Leontev derivatives $D^{n_p}f$ are analytic and univalent in Δ . We find upper bound for the Nachbin's growth parameter Ψ-type of an entire function in terms of the exponents n_p . The notion of Ψ-order of an entire function is introduced in this chapter and the characterization of the Ψ-order in terms of the Taylor coefficients of the function f is found. Using this coefficient characterization , an upper bound for Ψ-order of the function f in terms of the exponents n_p is obtained. The results of Shah [80] and Shah and Trimble [86] follow from the results found in this chapter.

In Chapter V, we introduce the class of G-L absolute starlike functions consisting of functions f for which $D^k f$, $k = 0, 1, 2, \ldots$ are analytic in Δ and the following system of inequalities are satisfied.

$$\sum_{n=2}^{\infty} n \frac{|D^{n+k} f(0)|}{d_1 \cdots d_n} \leq \frac{|D^{k+1} f(0)|}{d_1}, \quad k = 0, 1, 2, \ldots$$

If f is G-L absolute starlike, then $D^k f$, $k = 0, 1, 2, \ldots$ are univalent in Δ.

Analogously, the class of G-L absolute convex functions is introduced that consists of functions f such that $D^k f$, $k = 0, 1, 2, \ldots$ are analytic in Δ and the following system of inequalities are satisfied:

$$\sum_{n=2}^{\infty} n^2 \frac{|D^{n+k} f(0)|}{d_1 \cdots d_n} \leq \frac{|D^{k+1} f(0)|}{d_1}, \quad k = 0, 1, 2, \ldots$$

If f is G-L absolute convex function, then $D^k f$, $k = 0, 1, 2, \ldots$ are convex in Δ. We first develop some preliminary results concerning a generalization of Appell polynomials. Using these results, we find the sharp coefficient estimates for G-L absolute starlike and G-L absolute convex functions, when $D^k f$ are entire. In the process, two polynomial representations for entire functions of finite Ψ-type are found. Simplicial representation formulae for functions belonging to certain subclasses of G-L absolute starlike and G-L absolute convex functions are also found. Some of the results of Buckholtz and Shah [18] are special cases of the results found in this chapter.

Finally in the last chapter, motivated by the definitions of spiral-like functions and starlike functions with respect to N-symmetric points, we define certain classes of functions related to Gelfond-Leontev derivatives. The coefficient estimates of functions belonging to these subclasses are determined. Further, the effect of fixed second coefficient on the growth of other coefficients for functions in these classes is also studied. The results found in this chapter include several known results.

CHAPTER II

UNIVALENT FUNCTIONS WITH NAGATIVE COEFFICIENTS HAVING UNIVALENT GELFOND-LEONTEV DERIVATIVES

2.1 Let T denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n , \quad a_n \geq 0 \tag{2.1.1}$$

that are analytic and univalent in the unit disc $\Delta = \{z : |z| < 1\}$ and let C be the subclass of T consisting of convex functions in Δ. Let Df denotes the Gelfond-Leontev derivatives (c.f. Section 1.4) of f given by

$$Df(z) = d_1 - \sum_{n=2}^{\infty} d_n a_n z^{n-1}$$

where $\{d_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive numbers. For k = 2, 3, ... the kth derivative of a function f in T is given by

$$D^k f(z) = - \sum_{n=k}^{\infty} d_n \cdots d_{n-k+1} \, a_n z^{n-k} .$$

Denote $D^o f \equiv f$ and $D^1 f \equiv Df$.

In this chapter, we study subclasses of T consisting of functions f for which some or all the Gelfond-Leontev derivatives $D^k f$, k = 1, 2, ... are analytic and univalent in Δ . Analogous subclasses of C consisting of functions for which $D^k f$, k = 1, 2, ..., are analytic, univalent and convex in Δ are also studied. We begin with the following definitions .

Definition: 2.1.1 A function f in T is said to be in the class $T_1(D)$ if its Gelfond-Leontev derivatives Df is analytic and univalent in Δ .

With $d_n \equiv n$, the class $T_1(D)$ reduces to the class T_1 studied by Silverman [94]. However, we shall show in Section 2.2 that, in general, the class T_1 need not be contained in $T_1(D)$ and conversely $T_1(D)$ also need not be contained in T_1.

Definition 2.1.2 A function f in T is said to be in the class $C_1(D)$ if f is convex in Δ and its Gelfond-Leontev derivative Df is analytic, univalent and convex in Δ.

Let C_1 denote the class of functions in T for which f' is also convex in Δ. Note that the class $C_1(D)$ is a subclass of C (c.f. Section 1.2) and, for $d_n \equiv n$, $C_1(D)$ reduces to the class C_1. It will be seen in Section 2.2 that, in general, the class C_1 need not be contained in $C_1(D)$ and conversely $C_1(D)$ need not be contained in C_1.

Besides showing that, in general, the class T_1 need not be contained in $T_1(D)$ and conversely, the class $T_1(D)$ need not be contained in T_1, in Section 2.2, we obtain sufficient conditions on the coefficients for functions to be in the class $T_1(D)$. Similar results for the classes C_1 and $C_1(D)$ are also obtained in this section. Section 2.3 deals with the determination of some necessary conditions on the coefficients for contain polynomials in $T_1(D)$. The necessary conditions on the coefficients of certain polynomials in $C_1(D)$ are found in Section 2.4. In Section 2.5, coefficient bounds for functions in the class $T_1(D)$ which need not be polynomials are

obtained. Finally in Section 2.6, we find a sufficient condition on the coefficients for the univalent functions with negative coefficients to have higher order univalent Gelfond-Leontev derivatives or higher order convex Gelfond-Leontev derivatives.

2.2 In this section we find sufficient conditions on the coefficients of functions in T or C to be in the class $T_1(D)$ or $C_1(D)$. We begin by constructing examples to show that $T_1 \not\subset T_1(D)$ and, conversely, $T_1(D) \not\subset T_1$.

<u>Theorem</u> 2.2.1 <u>There exists a function</u> $F \in T$ <u>such that</u> F' <u>is univalent in</u> Δ <u>but its Gelfond-Leontev derivative</u> DF <u>is not univalent in</u> Δ <u>and, conversely, there exists a function</u> $G \in T$ <u>such that</u> DG <u>is univalent in</u> Δ <u>but its derivative</u> G' <u>is not univalent in</u> Δ.

<u>Proof</u>. To see the first part, let

$$a_n = \frac{a_2}{2^{n-3}n(n-1)}, \quad 0 < a_2 \leq \frac{1}{3}, \quad n = 3,4,\ldots, n \neq N+1$$

for some fixed positive integer N and

$$|a_{N+1}| = \frac{a_2}{2^{N-2}N(N+1)}.$$

Now, let $\{d_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive numbers such that $d_1 = 1$, $d_{N+1} = d_2\, 2^{N-2}(N+2)$, $d_{N+k} = d_{N+1}$, for all $k > 1$.

Then,

$$\frac{d_{N+1}|a_{N+1}|}{d_2 a_2} = \frac{N+2}{N(N+1)} > \frac{1}{N}.$$

Hence , by a theorem of Qin Yuan-Xun [65,30] , there exists a real number φ_o such that the related function

$$(2.2.1)\qquad z+\sum_{n=2}^{N-1} \frac{d_{n+1}a_{n+1}}{d_2a_2} z^n + e^{i\varphi_o} \frac{d_{N+1}a_{N+1}}{d_2a_2} z^N + \sum_{n=N+1}^{\infty} \frac{d_{n+1}a_{n+1}}{d_2a_2} z^n$$

is not univalent in Δ. Consider the function F , defined by

$$F(z) = z - \sum_{n=2}^{N} a_n z^n - e^{i\varphi_o} a_{N+1} z^{N+1} - \sum_{n=N+2}^{\infty} a_n z^n .$$

We can choose the argument of a_{N+1} such that $e^{i\varphi_o} a_{N+1} > 0$. Since,

$$\sum_{n=2}^{N} na_n + e^{i\varphi_o} (N+1) a_{N+1} + \sum_{n=N+2}^{\infty} na_n \leq 1$$

by (1.2.6), $F \in T$. Further , F' is also seen to be univalent in Δ. Now,

$$DF(z) = d_1 - d_2a_2z - \cdots - d_Na_Nz^{N-1} - e^{i\varphi_o} d_{N+1}a_{N+1}z^N - \cdots ,$$

In view of (2.2.1), DF is not univalent in Δ. This proves the first half of the theorem.

To see the second part , let

$$a_2 = \frac{d_3(d_2+d_4)}{\sqrt{2d_2+d_4}\,(3d_2+2d_3\sqrt{2d_2+d_4})} , \quad a_3 = \frac{d_2\sqrt{d_4}}{3d_2+2d_3\sqrt{2d_2+d_4}} .$$

and

$$|a_4| = \frac{d_2d_3}{2\sqrt{2d_2+d_4}\,(3d_2+2d_3\sqrt{2d_2+d_4})}$$

where the sequence $\{d_n\}$ of increasing positive numbers is

chosen such that $2d_2 > d_4$. Then, by the choice of d_n's

$$\frac{4|a_4|}{2a_2} = \frac{d_2}{d_2+d_4} > \frac{1}{3} .$$

Therefore, by a theorem of Qin Yuan-Xun [65, 30] there exists a real number φ_1 such that the related function

$$z + \frac{3a_3}{2a_2} z^2 + e^{i\varphi_1} \frac{2a_4}{a_2} z^3 \tag{2.2.2}$$

is not univalent in Δ. Consider the function G, defined by

$$G(z) = z - a_2 z^2 - a_3 z^3 - e^{i\varphi_1} a_4 z^4 .$$

We can choose the argument of a_4 such that $e^{i\varphi_1} a_4 > 0$. Since,

$$2a_2 + 3a_3 + 4e^{i\varphi_1} a_4 = 1$$

it follows by (1.2.6) that $G \in T$.

Now,

$$\frac{d_1 - DG(z)}{d_2 a_2} = z + \frac{\sqrt{d_4}}{\sqrt{2d_2+d_4}} z^2 + \frac{d_4}{2(d_2+d_4)} z^3 .$$

In view of (1.2.4), it is seen that $DG(z)$ is univalent in Δ.

However, G' is given by

$$G'(z) = 1 - 2a_2 z - 3a_3 z^2 - 4e^{i\varphi_1} a_4 z^3$$

so that by (2.2.2), $G'(z)$ is not univalent in Δ. This completes the proof of the theorem.

Our next theorem gives a sufficient condition for functions in T to be in the class $T_1(D)$.

Theorem 2.2.2. Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_2 > o$ be in T. If

$$\sum_{n=3}^{\infty} (n-1) d_n a_n \leq d_2 a_2 \tag{2.2.3}$$

then f is in $T_1(D)$.

Proof : Write,

$$g(z) = \frac{d_1 - Df(z)}{d_2 a_2} = z + \sum_{n=2}^{\infty} \frac{d_{n+1} a_{n+1}}{d_2 a_2} z^n. \tag{2.2.4}$$

$$= z + \sum_{n=2}^{\infty} b_n z^n \text{ (say)}.$$

The function $Df(z)$ is univalent in Δ, if and only if, $g(z)$ is in S. Since, by (1.2.16),

$$\sum_{n=2}^{\infty} nb_n = \sum_{n=2}^{\infty} n \frac{d_{n+1} a_{n+1}}{d_2 a_2} \leq 1$$

which is a sufficient condition for $g(z)$ to be in S, we have $Df(z)$ is univalent in Δ. Thus, $f \varepsilon T_1(D)$.

Remark. The inequality in (2.2.3) is sharp in the following sense : For given $\varepsilon > o$, there exists a sequence $\{a_n(\varepsilon)\}_{n=2}^{\infty}$ such that $\sum_{n=3}^{\infty} (n-1) d_n a_n = d_2 a_2 + \varepsilon$ and $F_\varepsilon(z) = z - \sum_{n=2}^{\infty} a_n(\varepsilon) z^n$ is in $T-T_1(D)$. To demonstrate this, let $o < a_2 < 1/2$ and n be any positive integer such that

$$2a_2 + (n+1) \frac{d_2 a_2 + \varepsilon}{n d_{n+1}} \leq 1. \tag{2.2.5}$$

Let

$$F_\varepsilon(z) = z - a_2 z^2 - \frac{d_2 a_2 + \varepsilon}{n d_{n+1}} z^{n+1}.$$

Clearly, $\sum_{n=3}^{\infty} (n-1)d_n a_n = d_2a_2+\varepsilon$ and , in view of (2.2.5) , $F_\varepsilon \in T$. But , $g_\varepsilon(z) = (d_1 - DF_\varepsilon(z))/d_2a_2$ is not in S since $g_\varepsilon'(z) = 1+(1+\varepsilon/d_2a_2)z^{n-1} = o$, for some z_o in Δ. Thus, $F_\varepsilon \notin T_1(D)$.

<u>Corollary</u> 2.2.1. <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq o$, $n = 3,4,\ldots,$ $o < a_2 \leq 2d_3/(4d_3+3d_2)$. <u>If</u> (2.2.3) <u>holds</u>, <u>then</u> f <u>is in</u> $T_1(D)$.

<u>Proof</u>. From (2.2.3), we have

$$\sum_{n=3}^{\infty} d_n a_n \leq \frac{d_2a_2}{2} . \tag{2.2.6}$$

Further, since $\{d_n\}_{n=1}^{\infty}$ is non-decreasing ,

$$d_3 \sum_{n=3}^{\infty} na_n \leq \sum_{n=3}^{\infty} (n-1)d_n a_n + \sum_{n=3}^{\infty} d_n a_n .$$

Now, using (2.2.3) and (2.2.6) in the above inequality, we get for $o < a_2 \leq 2d_3/(4d_3+3d_2)$,

$$\sum_{n=2}^{\infty} na_n \leq 2a_n + \frac{3d_2a_2}{2d_3} \leq 1.$$

This proves $f \in T$ and Corollary 2.2.1 follows from Theorem 2.2.2.

Taking $d_n \equiv 1$ in Corollary 2.2.1 , we get

<u>Corollary</u> 2.2.2 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq o$, $o < a_2 \leq 2/7$. <u>If</u>

$$\sum_{n=2}^{\infty} (n-1)a_n \leq a_2 \tag{2.2.7}$$

<u>then</u> f <u>and</u> Lf <u>are univalent in</u> Δ.

In the following theorem , a necessary and sufficient condition on the coefficients for a function $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ to be in the class $T_1(D)$ where , a_n's need not be real , is obtained.

<u>Theorem</u> 2.2.3 <u>Let</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n/a_2 \leq 0$, $n = 3, 4, \ldots$, $a_2 \neq 0$. <u>Then</u> Df <u>is univalent in</u> Δ, <u>if and only if</u>, $\sum_{n=3}^{\infty} (n-1) d_n |a_n| \leq d_2 |a_2|$. <u>Further</u>, <u>if</u> $|a_2| \leq 2d_3/(4d_3+3d_2)$, <u>then</u> f <u>belongs to</u> $T_1(D)$.

<u>Proof</u>. The function Df(z) is univalent in Δ, if and only if,

$$(2.2.8) \qquad g(z) = \frac{d_1 - Df(z)}{d_2 a_2} = z - \sum_{n=2}^{\infty} \frac{d_{n+1}|a_{n+1}|}{d_2|a_2|} z^n$$

$$= z - \sum_{n=2}^{\infty} b_n z^n \text{ (say)}$$

is in T. Since, $\sum_{n=2}^{\infty} nb_n \leq 1$ is a necessary and sufficient condition for g(z) to be in T, we see that the function Df(z) is univalent in Δ, if and only if, $\sum_{n=3}^{\infty} (n-1) d_n |a_n| \leq d_2 |a_2|$. This proves the first part of the theorem.

It is easily seen by following the same lines of proof as in Corollary 2.2.1 that, if $|a_2| \leq 2d_3/(4d_3+3d_2)$, in addition to $\sum_{n=3}^{\infty} (n-1) d_n |a_n| \leq d_2 |a_2|$, then f(z) is also in S. Thus, in this case, $f \in T_1(D)$.

Putting $d_n \equiv 1$ in Theorem 2.2.3, we get the following result

<u>Corollary</u> 2.2.3 <u>Let</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n/a_2 \leq 0$, $a_2 \neq 0$. <u>Then</u>, Lf <u>is univalent in</u> Δ, <u>if and only if</u>, $\sum_{n=3}^{\infty} (n-1)/a_n| \leq |a_2|$. <u>Further</u>, <u>if</u> $|a_2| \leq 2/7$, <u>then both</u> f <u>and</u> Lf <u>are univalent in</u> Δ.

<u>Remark</u>. By taking $d_n \equiv n$ some of the results of Silverman [94] follow from Theorems 2.2.2, 2.2.3 and Corollary 2.2.1.

Now we give an example to show that $C_1 \not\subset C_1(D)$ and, conversely, $C_1(D) \not\subset C_1$.

Theorem 2.2.4 *There exists a function F in C such that F' is convex in Δ but its Gelfond-Leontev derivative DF is not convex in Δ. Conversely, there exists a function G such that its Gelfond-Leontev derivative DG is convex in Δ but its derivative G' is not convex in Δ.*

Proof. Choose

$$a_n = \frac{a_2}{2^{n-3}n^2(n-1)}, \quad 0 < a_2 \leq \frac{1}{5}, \; n=3,4,\ldots, n \neq N+1$$

for some fixed positive integer $N \geq 3$ and

$$|a_{N+1}| = \frac{a_2}{2^{N-2}N(N+1)^2}.$$

Let $\{d_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive numbers such that

$$d_1 = 1, \quad d_{N+1} = d_2 2^{N-2}(N+2)^2, \quad d_{N+k} = d_{N+1} \text{ for all } k > 1.$$

Since,

$$\frac{d_{N+1}|a_{N+1}|}{d_2 a_2} > \frac{1}{N}$$

it follows [65, 30] that there exists a real number φ_0 such that the function

$$(2.2.9) \qquad z + \frac{d_3 a_3}{d_2 a_2} z^2 + \ldots + e^{i\varphi_0} \frac{d_{N+1} a_{N+1}}{d_2 a_2} z^N + \ldots$$

is not univalent in Δ. Consider the function F, defined by

$$F(z) = z - a_2 z^2 - a_3 z^3 - \ldots - a_N z^N - e^{i\varphi_0} a_{N+1} z^{N+1} - \ldots$$

We choose the argument of a_{N+1} such that $e^{i\varphi_o} a_{N+1} > o$. Since

$$\sum_{n=2}^{N} n^2 a_n + (N+1)^2 e^{i\varphi_o} a_{N+1} + \sum_{n=N+2}^{\infty} n^2 a_n \leq 1$$

it follows by (1.2.28) that $F \in C$. It is easily seen that F' is also convex in Δ. Now,

$$DF(z) = d_1 - d_2 a_2 z - \cdots - d_N a_N z^{N-1} - e^{i\varphi_o} d_{N+1} a_{N+1} z^N - \cdots$$

In view of (2.2.9), the function DF is not univalent in Δ and hence not convex in Δ. This completes the proof of first half of the theorem.

To show the converse part, let

$$a_2 = \frac{1}{18A}, \quad |a_3| = \frac{d_2\sqrt{\frac{2t_o(1-t_o)}{27-25t_o}}}{9d_3 A}, \quad a_4 = \frac{d_2 t_o a_2}{9d_4},$$

where

$$A = \frac{2}{9}\left(1+\frac{4d_2}{9d_4}t_o\right) + \frac{d_2}{d_3}\sqrt{\frac{2t_o(1-t_o)}{27-25t_o}}, \quad t_o = (27-3\sqrt{6})/25.$$

Let the positive numbers d_2 and d_3 be such that

$$\text{(2.2.10)} \qquad \frac{3d_2}{d_3}\sqrt{\frac{2t_o(1-t_o)}{27-25t_o}} > \frac{1}{2}.$$

Due to (2.2.10),

$$\frac{3|a_3|}{2a_2} = \frac{3d_2\sqrt{\frac{2t_o(1-t_o)}{27-25t_o}}}{d_3} > \frac{1}{2}.$$

Therefore, it follows [65, 30] that there is a real number φ_1 such that the related function

$$z + e^{i\varphi_1} \frac{3a_3}{2a_2} z^2 + \frac{2a_4}{a_2} z^3 \tag{2.2.11}$$

is not univalent in Δ. Consider the function

$$G(z) = z-a_2z^2-a_3e^{i\varphi_1} z^3-a_4z^4 .$$

We choose the argument of a_3 such that $a_3e^{i\varphi_1} > o$. Since,

$$4a_2+9a_3e^{i\varphi_1} +16a_4 = 1$$

we have by (1.2.28), the function G is in C. Further, in view of (1.2.12) the function

$$\frac{d_1-DG(z)}{d_2a_2} = z + \frac{d_3a_3e^{i\varphi_1}}{d_2a_2}z^2 + \frac{d_4a_4}{d_2a_2} z^3$$

is convex in Δ. But, since

$$G'(z) = 1-2a_2z-3a_3e^{i\varphi_1} z^2-4a_4z^3$$

in view of (2.2.11) $G'(z)$ is not univalent in Δ and hence not convex in Δ. Hence the theorem.

In the next theorem, we find a sufficient condition on the coefficients for functions in C to be in the class $C_1(D)$.

<u>Theorem</u> 2.2.5 <u>Let</u> $f(z) = z-\sum_{n=2}^{\infty} a_nz^n$, $a_2 > o$, <u>be in</u> C. <u>If</u>,

$$\sum_{n=3}^{\infty} (n-1)^2d_na_n \leq d_2a_2 \tag{2.2.12}$$

<u>then</u> f <u>belongs to the class</u> $C_1(D)$.

Proof. We have, the function $Df(z)$ is convex in Δ, if and only if, the function $g(z)$ defined by (2.2.4) is in K. Since, $\sum_{n=2}^{\infty} n^2 b_n = \sum_{n=2}^{\infty} n^2 \frac{d_{n+1} a_{n+1}}{d_2 a_2} \leq 1$ is a sufficient condition for $g(z)$ to be in K, we have $Df(z)$ is convex in Δ. Thus, $f \in C_1(D)$.

With $d_n \equiv n$. Theorem 2.2.5 gives the following sufficient condition for functions in C to be in the class C_1.

Corollary 2.2.4 *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_2 > 0$, *be in* C. *If*

$$\sum_{n=3}^{\infty} n(n-1)^2 a_n \leq 2a_2 \tag{2.2.13}$$

then f belongs to C_1.

Taking $d_n \equiv 1$ in Theorem 2.2.5, we get

Corollary 2.2.5 *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_2 > 0$, *be in* C. *If*

$$\sum_{n=3}^{\infty} (n-1)^2 a_n \leq a_2 \tag{2.2.14}$$

then $Lf(z)$ *is convex in* Δ.

Corollary 2.2.6 *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, $n = 3, 4, \ldots$, $0 < a_2 \leq 4d_3/(16d_3 + 9d_2)$. *If* (2.2.12) *holds, then* f *belongs to* $C_1(D)$.

Proof. In view of (2.2.12), it is sufficient to show that $f \in C$. Since $((n-1)/n)^2 d_n$ is a non-decreasing function of n, it follows that for $n = 3, 4, \ldots$

$$\left(\frac{n-1}{n}\right)^2 d_n \geq \frac{4}{9} d_3$$

Therefore,

$$\tfrac{4}{9}d_3 \sum_{n=3}^{\infty} n^2 a_n \leq \sum_{n=3}^{\infty} (n-1)^2 d_n a_n \leq d_2 a_2 .$$

Since, for $o < a_2 \leq 4d_3/(16d_3+9d_2)$,

$$\sum_{n=2}^{\infty} n^2 a_n \leq 4a_2 + \frac{9d_2}{4d_3} a_2 \leq 1 .$$

it follows by (1.2.28) that $f \varepsilon C$. This proves Corollary 2.2.6.

Putting $d_n \equiv n$ in Corollary 2.2.6, we get

<u>Corollary</u> 2.2.7 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq o$, $o < a_2 \leq 2/11$. <u>If</u> (2.2.13) holds, <u>then</u> f <u>belongs</u> <u>to</u> C_1.

Taking $d_n \equiv 1$ in Corollary 2.2.6, we have

<u>Corollary</u> 2.2.8 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq o$, $a < a_2 \leq 4/25$. <u>If</u> (2.2.14) <u>holds</u>, <u>then</u> <u>both</u> f <u>and</u> Lf <u>are</u> <u>convex</u> <u>in</u> Δ.

In the following theorem, we find necessary and sufficient condition on the coefficients for functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ to be in the class $C_1(D)$ where a_n's need not be real.

<u>Theorem</u> 2.2.6 <u>Let</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n/a_2 \leq o$, $a_2 \neq o$. <u>Then</u> Df <u>is</u> <u>convex</u> <u>in</u> Δ, <u>if</u> <u>and</u> <u>only</u> <u>if</u>, $\sum_{n=3}^{\infty} (n-1)^2 d_n |a_n| \leq d_2 |a_2|$. <u>Further</u>, <u>if</u> $|a_2| \leq 4d_3/(16d_3+9d_2)$, <u>then</u> <u>both</u> f <u>and</u> Df <u>are</u> <u>convex</u> <u>in</u> Δ.

<u>Proof</u>. Since, $\sum_{n=2}^{\infty} n^2 b_n \leq 1$ is a necessary and sufficient condition for a function $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, defined by (2.2.8), to be in C, the result follows by using the same lines of proof as in Theorem 2.2.3.

With $d_n \equiv n$, Theorem 2.2.6 gives

Corollary 2.2.9 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n/a_2 \leq 0$, $a_2 \neq 0$. Then f' is convex in Δ, if $\sum_{n=3}^{\infty} n(n-1)^2 a_n \leq 2|a_2|$. Further, if $|a_2| \leq 2/11$, then both f and f' are convex in Δ.

Putting $d_n \equiv 1$ in Theorem 2.2.6, we get

Corollary 2.2.10 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n/a_2 \leq 0$, $a_2 \neq 0$. Then $Lf(z)$ is convex in Δ, if and only if, $\sum_{n=3}^{\infty} n(n-1)^2 |a_n| \leq |a_2|$. Further, if $|a_2| \leq 4/25$, then both f and Lf are convex in Δ.

2.3 We find in this section some sharp bounds on the coefficients which are necessarily satisfied for certain polynomials in $T_1(D)$.

Theorem 2.3.1. Let $f(z) = z - a_2 z^2 - a_{p+1} z^{p+1}$, $p \geq 2$, ($a_2 > 0$, $a_{p+1} \geq 0$). Then f belongs to $T_1(D)$, if and only if,

$$(2.3.1) \qquad a_{p+1} \leq \min \begin{cases} \dfrac{1-2a_2}{p+1} \\ \\ \dfrac{d_2}{pd_{p+1}} a_2 \end{cases}$$

Proof. We have (c.f. section 1.2)

$$(2.3.2) \qquad (p+1) a_{p+1} \leq 1 - 2a_2$$

if and only if, $f \in T$. On the otherhand, the function Df is

univalent in Δ, if and only if $g(z) = (d_1 - Df(z))/d_2 a_2$
$= z + (d_{p+1} a_{p+1}/d_2 a_2) z^p \in S$. But, a necessary and sufficient condition for $g(z)$ to be in S is

$$\frac{d_{p+1} a_{p+1}}{d_2 a_2} \leq \frac{1}{p} . \tag{2.3.3}$$

Now, (2.3.2) and (2.3.3) hold, if and only if, $f \in T_1(D)$. Hence the theorem.

<u>Corollary</u> 2.3.1 <u>Let</u> $f(z) = z - a_2 z^2 - a_{p+1} z^{p+1}$, $p \geq 2 (a_2 > 0, a_{p+1} \geq 0)$ be in $T_1(D)$. <u>Then</u>,

$$a_{p+1} \leq \frac{d_2}{(p+1)d_2 + 2pd_{p+1}} . \tag{2.3.4}$$

<u>Equality in</u> (2.3.4) <u>holds, if and only if</u>,

$$F(z) = z - \frac{pd_{p+1}}{(p+1)d_2 + 2pd_{p+1}} z^2 - \frac{d_2}{(p+1)d_2 + 2pd_{p+1}} z^{p+1} .$$

<u>Proof</u>. Since $f \in T_1(D)$, we have by (2.3.1)

$$a_{p+1} \leq \frac{d_2 a_2}{pd_{p+1}} \leq \frac{d_2}{pd_{p+1}} \frac{(1-(p+1)a_{p+1})}{2} .$$

This implies that $a_{p+1} \leq d_2/((p+1)d_2 + 2pd_{p+1})$. Equality at both the ends holds if and only if $a_2 = pd_{p+1}/((p+1)d_2 + 2pd_{p+1})$ and $a_{p+1} = d_2/((p+1)d_2 + 2pd_{p+1})$.

Theorem 2.3.1 shows that the function $F(z)$ is in $T_1(D)$.

<u>Remark</u>. For $d_n \equiv n$ and $p = 2$, we get a result of Silverman [94] from Corollary 2.3.1.

With $d_n \equiv 1$ in Corollary 2.3.1, we get the following result for shift operator.

$$(2.3.7) \qquad a_3 \leq \begin{cases} \dfrac{d_2 a_2 + 3d_4 a_4}{2d_3}, & o \leq a_4 \leq \dfrac{d_2}{5d_4} a_2 \\[2ex] \dfrac{2\sqrt{d_4 a_4 (d_2 a_2 - d_4 a_4)}}{d_3}, & \dfrac{d_2}{5d_4} a_2 \leq a_4 \leq \dfrac{d_2}{3d_4} a_2 \end{cases}$$

Further, f is in T, if and only if,

$$(2.3.8) \qquad a_3 \leq \frac{1-2a_2-4a_4}{3}$$

Thus, from (2.3.7) and (2.3.8), $f \varepsilon T_1(D)$, if and only if, for $o \leq a_4 \leq (d_2/5d_4)a_2$

$$(2.3.9) \qquad a_3 \leq \min \begin{cases} \dfrac{d_2 a_2 + 3d_4 a_4}{2d_3} \\[2ex] \dfrac{1-2a_2-4a_4}{3} \end{cases}$$

and, for $(d_2/5d_4)a_2 \leq a_4 \leq (d_2/3d_4)a_2$,

$$(2.3.10) \qquad a_3 \leq \min \begin{cases} \dfrac{2\sqrt{d_4 a_4 (d_2 a_2 - d_4 a_4)}}{d_3} \\[2ex] \dfrac{1-2a_2-4a_4}{3}. \end{cases}$$

The right side of (2.3.9) attains its maximum at a point where

$$\frac{d_2 a_2 + 3d_4 a_4}{2d_3} = \frac{1-2a_2-4a_4}{3}$$

or

(2.3.11)
$$(4d_3+3d_2)a_2 + (9d_4+8d_3)a_4 = 2d_3$$

Setting $a_4 \equiv a_4(t) = \frac{d_2}{5d_4} ta_2$, $o \leq t \leq 1$, (2.3.11) gives

$$a_2 = a_2(t) = \frac{2d_3}{[(4d_3+3d_2)+\frac{d_2}{5d_4}(9d_4+8d_3)t]}$$

Let ,

$$h_1(t) = \frac{d_2a_2(t)+3d_4a_4(t)}{2d_3}$$

Since , for $o \leq t \leq 1$

$$h_1'(t) = \frac{4d_2d_3(3d_4-2d_2)}{5d_4[(4d_3+3d_2)+\frac{d_2}{5d_4}(9d_4+8d_3)t]^2} > o$$

$h_1(t)$ is an increasing function of t . Therefore , $h_1(t) \leq h_1(1) = (4d_2/5d_3)a_2(1)$. By (2.3.9), this gives for $o \leq a_4 \leq (d_2/5d_4)a_2$

(2.3.12)
$$a_3 \leq \frac{8d_2}{5[(4d_3+3d_2)+\frac{d_2}{5d_4}(9d_4+8d_3)]}$$

Similarly , the right hand side of (2.3.10) is maximized at a point where

(2.3.13)
$$\frac{2}{d_3}\sqrt{d_4a_4(d_2a_2-d_4a_4)} = \frac{1-2a_2-4a_4}{3}$$

Let $a_4 = \tilde{a}_4(t) = \frac{d_2}{3d_4} t a_2$, $\frac{3}{5} \leq t \leq 1$. Then , from (2.3.13) , we obtain

$$a_2 = \tilde{a}_2(t) = \frac{d_3}{2[d_2\sqrt{(3-t)t} + d_3(1+\frac{2d_2}{3d_4}t)]}$$

Let ,

$$h_2(t) = \frac{1-2\tilde{a}_2(t)-4\tilde{a}_4(t)}{3} = \frac{1}{3}[1-2(1+\frac{2d_2}{3d_4}t)\tilde{a}_2(t)]$$

Then ,

$$h_2'(t) = \frac{d_2d_3[3d_4-2(d_2+d_4)t]}{3d_4\sqrt{(3-t)t}\,[d_2\sqrt{(3-t)t}+d_3(1+\frac{2d_2}{3d_4}t)]^2}$$

Now two cases arise.

(i) $d_4 \geq 2d_2$. In this case , $h_2'(t) \geq 0$, $3/5 \leq t \leq 1$. Thus , $h_2(t)$ is an increasing function of t . So , $h_2(t) \leq h_2(1) = [1-2(1+2d_2/3d_4)\tilde{a}_2(1)]/3$. Therefore , by (2.3.13) , for $(d_2/5d_4)a_2 \leq a_4 \leq (d_2/3d_4)a_2$

$$\text{(2.3.14)} \qquad a_3 \leq \frac{d_2\sqrt{2}}{3[d_2\sqrt{2}+d_3(1+\frac{2d_2}{3d_4})]}$$

Since , for $d_4 \geq 2d_2$,

$$\frac{d_2\sqrt{2}}{3[d_2\sqrt{2}+d_3(1+\frac{2d_2}{3d_4})]} > \frac{8d_2}{5[(4d_3+3d_2)+\frac{d_2}{5d_4}(9d_4+8d_3)]}$$

we have by (2.3.12) and (2.3.14) , for $0 \leq a_4 \leq (d_2/3d_4)a_2$

$$\text{(2.3.15)} \qquad a_3 \leq \frac{d_2\sqrt{2}}{3[d_2\sqrt{2} + d_3(1+\frac{2d_2}{3d_4})]}$$

(ii) $d_4 \leqq 2d_2$. In this case, we observe that $h_2''(t_o) < o$ for $t_o = 3d_4/2(d_4+d_2)$. Further, $3/5 \leqq t_o \leqq 1$. Therefore, $h_2(t)$ attains its maximum at t_o. This gives $h_2(t) \leqq h_2(3d_4/2(d_4+d_2))$ that is, by (2.3.13)

$$a_3 \leqq \frac{d_2\sqrt{d_4}}{3d_2\sqrt{d_4}+2d_3\sqrt{2d_2+d_4}}. \tag{2.3.16}$$

Since, for $d_4 \leqq 2d_2$,

$$\frac{d_2\sqrt{d_4}}{3d_2\sqrt{d_4}+2d_3\sqrt{2d_2+d_4}} > \frac{8d_2}{5[(4d_3+3d_2)+\frac{d_2}{5d_4}(9d_4+8d_3)]}$$

we have for $o \leqq a_4 \leqq (d_2/3d_4)a_2$

$$a_3 \leqq \frac{d_2\sqrt{d_4}}{3d_2\sqrt{d_4}+2d_3\sqrt{2d_2+d_4}} \tag{2.3.17}$$

This proves the inequality (2.3.6).

We note that,

$$a_2 = a_2(1) = \frac{d_3}{2[d_2\sqrt{2}+d_3(1+\frac{2d_2}{3d_4})]}; \quad a_4 = a_4(1) = \frac{d_2d_3}{6d_4[d_2\sqrt{2}+d_3(1+\frac{2d_2}{3d_4})]}$$

and

$$\tilde{a}_2 = \tilde{a}_2(t_o) = \frac{d_3(d_2+d_4)}{3d_2\sqrt{2d_2+d_4}+2d_3(2d_2+d_4)},$$

$$\tilde{a}_4 = \tilde{a}_4(t_o) = \frac{d_2d_3}{2[3d_2\sqrt{d_4(2d_2+d_4)}+2d_3(2d_2+d_4)]}.$$

Clearly , for $d_4 \leq 2d_2$, the function F_1 given by

$$F_1(z) = z - \frac{d_3(d_2+d_4)}{3d_2\sqrt{d_4(2d_2+d_4)}+2d_3(2d_2+d_4)}z^2 - \frac{d_2\sqrt{d_4}}{3d_2\sqrt{d_4}+2d_3\sqrt{2d_2+d_4}}z^3$$

$$- \frac{d_2d_3}{2[3d_2\sqrt{d_4(2d_2+d_4)}+2d_3(2d_2+d_4)]}z^4$$

and for $d_4 \geq 2d_2$, the function F_2 given by

$$F_2(z) = z - \frac{d_3}{2[d_2\sqrt{2}+d_3(1+\frac{2d_2}{3d_4})]}z^2 - \frac{d_2\sqrt{2}}{3[d_2\sqrt{2}+d_3(1+\frac{2d_2}{3d_4})]}z^3$$

$$- \frac{d_2d_3}{6d_4[d_2\sqrt{2}+d_3(1+\frac{2d_2}{3d_4})]}z^4$$

shows that the estimate in (2.3.6) is sharp. It is obvious by (1.2.4) that the functions $F_1(z)$ and $F_2(z)$ are in $T_1(D)$.

Remark. For $d_n \equiv n$, Theorem 2.3.2 gives a result of Silverman [94 , Theorem 4].

Taking $d_n \equiv 1$ in Theorem 2.3.2 , we get

Corollary : 2.3.3 *Let* $f(z) = z - a_2z^2 - a_3z^3 - a_4z^4$, $(a_2 > 0, a_3 \geq 0, a_4 \geq 0)$ *be in* T. *If* Lf *is univalent in* Δ , *then*

(2.3.18) $$a_3 \leq \frac{1}{3+2\sqrt{3}}$$

The result is sharp , with equality for the function

$$F(z) = z - \frac{2}{\sqrt{3}(3+2\sqrt{3})}z^2 - \frac{1}{(3+2\sqrt{3})}z^3 - \frac{1}{6(2+\sqrt{3})}z^4.$$

The next theorem gives a necessary condition satisfied by one of the coefficient of any sixth degree polynomial belonging to the class $T_1(D)$.

<u>Theorem</u> 2.3.3 <u>Let</u> $f(z) = z-a_2z^2-a_4z^4-a_6z^6, (a_2 > o, a_4 \geq o, a_6 \geq o)$ <u>be in</u> $T_1(D)$. <u>If</u> $d_6 \geq 3d_2$, <u>then</u>

(2.3.19)
$$a_4 \leq \frac{3d_2d_6}{10d_4d_6+12d_2d_6+6d_2d_4}.$$

<u>The estimate in</u> (2.3.19) <u>is sharp</u>.

<u>Proof</u>. The function $Df(z)$ is univalent in Δ, if and only if,

$$g(z) = \frac{d_1-Df(z)}{d_2a_2}$$
$$= z + \frac{d_4a_4}{d_2a_2}z^3 + \frac{d_6a_6}{d_2a_2}z^5$$

is in S. Equivalently, by (1.2.5), $Df(z)$ is univalent in Δ, if and only if,

(2.3.20)
$$a_4 \leq \begin{cases} \dfrac{d_2a_2+5d_6a_6}{3d_4}, & o \leq a_6 \leq \dfrac{d_2}{10d_6}a_2 \\[2ex] \dfrac{2\sqrt{d_6a_6(d_2a_2-d_6a_6)}-d_6a_6}{d_4}, & \dfrac{d_2}{10d_6}a_2 \leq a_6 \leq \dfrac{d_2}{5d_6}a_2. \end{cases}$$

Again, $f \in T$, if and only if,

(2.3.21)
$$a_4 \leq \frac{1-2a_2-6a_6}{4}.$$

Now, from (2.3.20) and (2.3.21), $f \in T_1(D)$, if and only if, for $o \leq a_6 \leq (d_2/10d_6)a_2$

$$(2.3.22) \quad a_4 \leq \min \begin{cases} \dfrac{d_2a_2+5d_6a_6}{3d_4} \\ \\ \dfrac{1-2a_2-6a_6}{4} \end{cases}$$

and, for $(d_2/10d_6)a_2 \leq a_6 \leq (d_2/5d_6)a_2$

$$(2.3.23) \quad a_4 \leq \min \begin{cases} \dfrac{2\sqrt{d_6a_6(d_2a_2-d_6a_6)}-d_6a_6}{d_4} \\ \\ \dfrac{1-2a_2-6a_6}{4} \end{cases}$$

The right side of (2.3.22) is maximized at a point where

$$\frac{d_2a_2+5d_6a_6}{3d_4} = \frac{1-2a_2-6a_6}{4}.$$

i.e., where

$$(2.3.24) \qquad 2(2d_2+3d_4)a_2+2(10d_6+9d_4)a_6 = 3d_4.$$

Set, $a_6=a_6(t) = \dfrac{d_2}{10d_6}ta_2$, $o \leq t \leq 1$. Then, (2.3.24) yields

$$a_2 = a_2(t) = \frac{3d_4}{\left[2(2d_2+3d_4)+\dfrac{d_2}{5d_6}(10d_6+9d_4)t\right]}$$

Consider the function, defined on $[o,1]$ by

$$h_1(t) = \frac{d_2a_2(t)+5d_6a_6(t)}{3d_4}$$

$$= \frac{d_2(1+\frac{t}{2})}{\left[2(2d_2+3d_4)+\dfrac{d_2}{5d_6}(10d_6+9d_4)t\right]}$$

Since ,

$$h_1'(t) = \frac{3d_2d_4 - \dfrac{9d_2^2d_4}{5d_6}}{\left[2(2d_2+3d_4)+\dfrac{d_2}{5d_6}(10d_6+9d_4)t\right]^2} > 0 ,$$

$h_1'(t)$ is an increasing function of t . Therefore ,

$$h_1(t) \leqq h_1(1) = \frac{3d_2}{2\left[2(2d_2+3d_4)+\dfrac{d_2}{5d_6}(10d_6+9d_4)\right]}$$

so that by (2.3.22), for $o \leqq a_6 \leqq (d_2/10d_6)a_2$

$$(2.3.25) \qquad a_4 \leqq \frac{3d_2}{2\left[2(2d_2+3d_4)+\dfrac{d_2}{5d_6}(10d_6+9d_4)\right]} .$$

Likewise , the right side of (2.3.23) is maximum at a point for which

$$(2.3.26) \qquad \frac{2\sqrt{d_6a_6(d_2a_2-d_6a_6)}-d_6a_6}{d_4} = \frac{1-2a_2-6a_6}{4} .$$

Setting $a_6 = \tilde{a}_6(t) = (d_2/5d_6)ta_2$, $1/2 \leqq t \leqq 1$, we see that (2.3.26) gives

$$a_2 = \tilde{a}_2(t) = \frac{d_4}{\left[2d_4+\dfrac{8d_2}{5}\sqrt{t(5-t)}+\dfrac{d_2}{5d_6}(6d_4-4d_6)t\right]} .$$

Let,

$$h_2(t) = \frac{1-2\tilde{a}_2(t)-6\tilde{a}_6(t)}{4}$$

$$= \frac{1-2\tilde{a}_2(t)(1+\dfrac{3d_2}{5d_6}t)}{4} .$$

Then ,

$$4h_2'(t) = \tilde{a}_2(t) \frac{\frac{d_2}{5d_6}\left[20d_6-(12d_2+8d_6)t-4d_6\sqrt{t(5-t)}\right]}{\sqrt{t(5-t)}\left[2d_4+\frac{8d_2}{5}\sqrt{t(5-t)}+\frac{d_2}{5d_6}(6d_4-4d_6)t\right]}.$$

Since for $d_6 \geqq 3d_2$, $h_2'(t) \geqq o$, we note that $h_2(t)$ increases with t . Thus , for $1/2 \leqq t \leqq 1$

$$h_2(t) \leqq h_2(1) = \frac{3d_2}{5\left[2d_4+\frac{16d_2}{5}+\frac{d_2}{5d_6}(6d_4-4d_6)\right]}$$

$$= \frac{3d_2d_6}{(10d_4d_6+12d_2d_6+6d_2d_4)} .$$

So that by (2.3.23) , for $(d_2/10d_6)a_2 \leq a_6 \leq (d_2/5d_6)a_2$

(2.3.27) $$a_4 \leq \frac{3d_2d_6}{10d_4d_6+12d_2d_6+6d_2d_4} .$$

For $d_6 \geqq 3d_2$, we observe that

$$\frac{3d_2d_6}{[10d_4d_6+12d_2d_6+6d_2d_4]} > \frac{3d_2}{2\left[2(2d_2+3d_4)+\frac{d_2}{5d_6}(6d_4-4d_6)\right]} .$$

Therefore, for $o \leq a_6 \leq (d_2/5d_6)a_2$,

$$a_4 \leq \frac{3d_2d_6}{(10d_4d_6+12d_2d_6+6d_2d_4)} .$$

This proves the estimate in(2.3.19). It is readily seen by use of (1.2.5) that the estimate in(2.3.19) is sharp for the function

$$F(z) = z - \frac{d_4}{\left[2d_4+\frac{16d_2}{5}+\frac{d_2}{5d_6}(6d_4-4d_6)\right]} z^2 - \frac{3d_2d_6}{[10d_4d_6+12d_2d_6+6d_2d_4]} z^4$$

$$- \frac{d_2d_4}{5d_6\left[2d_4+\frac{d_2}{5d_6}(6d_4-4d_6)+\frac{16d_2}{5}\right]} z^6 .$$

With $d_n = n$, $n = 1, 2, \ldots,$ in Theorem 2.3.3, we get the following result.

Corollary 2.3.4 Let $f(z) = z-a_2z^2-a_4z^4-a_6z^6 (a_2 > 0, a_4 \geq 0, a_6 \geq 0)$ be in T_1. Then,

(2.3.28) $$a_4 \leq \frac{1}{12} .$$

The estimate in (2.3.28) is sharp, with equality for the function $F(z) = z - \frac{5}{18}z^2 - \frac{1}{12}z^4 - \frac{1}{54}z^6$.

We now prove

Theorem 2.3.4 Let $f(z) = z-a_2z^2-a_3z^3-a_nz^n - \frac{d_2}{nd_{n+1}}a_2z^{n+1}$, $n \geq 4$, $(a_2 > 0, a_3 \geq 0, a_n \geq 0)$ be in $T_1(D)$. Then,

(2.3.29) $$a_3 \leq \frac{n(n-1)d_2d_nd_{n+1}A_{2,1}}{(n-1)d_3d_n(2nd_{n+1}+(n+1)d_2)+nd_2d_{n+1}\;A_{2,1}(3(n-1)d_n+2nd_3)}$$

and

(2.3.30) $$a_n \leq \frac{2nd_2d_3d_{n+1}A_{2,1}}{(n-1)d_3d_n(2nd_{n+1}+(n+1)d_2)+nd_2d_{n+1}A_{2,1}(3(n-1)+2nd_3)}$$

for $n = 4, 5, \ldots,$ and where $A_{2,1} = (n-1)\sin(2\pi/(n+1))/n\sin(\pi/(n+1))$.

The estimates in (2.3.29) and (2.3.30) are sharp.

Proof. Since the function $Df(z)$ is univalent in Δ, we have

$$g(z) = \frac{d_1 - Df(z)}{d_2 a_2}$$

$$= z + \frac{d_3 a_3}{d_2 a_2} z^2 + \frac{d_n a_n}{d_2 a_2} z^{n-1} + \frac{1}{n} z^n$$

is in S. By using (1.2.6), we get

$$(2.3.31) \qquad a_3 \leq \frac{d_2 A_{2,1}}{d_3} a_2$$

and

$$(2.3.32) \qquad a_n = \frac{2d_3}{(n-1)d_n} a_3 .$$

Again, for f in T, we must have

$$(2.3.33) \qquad 2a_2 + 3a_3 + na_n + \frac{(n+1)d_2 a_2}{nd_{n+1}} \leq 1.$$

Since $f \in T_1(D)$, (2.3.31), (2.3.32) and (2.3.33) are satisfied. Thus, using (2.3.32), (2.3.33), we get

$$\frac{2nd_{n+1} + (n+1)d_2}{nd_{n+1}} a_2 + \frac{3(n-1)d_n + 2nd_3}{(n-1)d_n} a_3 \leq 1$$

or

$$(2.3.34) \qquad a_2 \leq \frac{nd_{n+1}}{2nd_{n+1} + (n+1)d_2} \left(1 - \frac{3(n-1)d_n + 2nd_3}{(n-1)d_n} a_3\right)$$

Using the above inequality in (2.3.31), we deduce that

$$a_3 \leq \frac{n(n-1)d_2 d_n d_{n+1} A_{2,1}}{(n-1)d_3 d_n (2nd_{n+1} + (n+1)d_2) + nd_2 d_{n+1} A_{2,1}(3(n-1)d_n + 2nd_3)}$$

and by (2.3.32)

$$a_n \leq \frac{2nd_2d_3d_{n+1}\ A_{2,1}}{(n-1)d_3d_n(2nd_{n+1}+(n+1)d_2)+nd_2d_{n+1}A_{2,1}(3(n-1)d_n+2nd_3)}$$

Consider the function F, given by

$$F(z) = z-a_2z^2-a_3z^3-a_nz^n - \frac{d_2}{nd_{n+1}}\ a_2z^{n+1}\ , \text{ where}$$

$$a_2 = \frac{n(n-1)d_3d_nd_{n+1}}{B_n}$$

$$a_3 = \frac{n(n-1)d_2d_nd_{n+1}A_{2,1}}{B_n}.$$

$$a_n = \frac{2nd_2d_3d_{n+1}A_{2,1}}{B_n}$$

and

$$B_n = (n-1)d_3d_n(2nd_{n+1}+(n+1)d_2)+nd_2d_{n+1}A_{2,1}(3(n-1)d_n+2nd_3).$$

Since $2a_2+3a_3+na_n+(n+1)\ \frac{d_2a_2}{nd_{n+1}} = 1$ and the function

$$\frac{d_1-DF(z)}{d_2a_2} = z+A_{2,1}\,z^2+A_{n-1,1}\,z^{n-1}+\frac{1}{n}z^n$$

is univalent in Δ [102], the function $F(z)$ shows that the estimate in (2.3.29) and (2.3.30) are sharp.

Putting $d_k \equiv k$, and $n = 4$ in Theorem 2.3.4, we get the following necessary condition satisfied by certain coefficients of the fifth degree polynomial belonging to the class T_1.

Corollary 2.3.5 Suppose $f(z) = z-a_2z^2-a_3z^3-a_4z^4-\frac{a_2}{10}z^5$ $(a_2>0,$ $a_3\geq 0, a_4\geq 0)$ is in T_1. Then ,

(2.3.35) $$a_3\leq \frac{\sqrt{5}-1}{10}$$

and

(2.3.36) $$a_4\leq \frac{\sqrt{5}-1}{20}$$

The estimate in (2.3.35) and (2.3.36) are sharp for the function

$$F(z) = z-\frac{(\sqrt{5}-1)^2}{10}z^2-\frac{\sqrt{5}-1}{10}z^3-\frac{\sqrt{5}-1}{20}z^4-\frac{(\sqrt{5}-1)^2}{100}z^5 .$$

With $d_k \equiv 1$ and $n = 4$, Theorem 2.3.4 gives

Corollary 2.3.6 Let $f(z) = z-a_2z^2-a_3z^3-a_4z^4-\frac{a_2}{4}z^5$ $(a_2>0, a_3\geq 0,$ $a_4\geq 0)$. If f and Lf are univalent in Δ , then

(2.3.37) $$a_3\leq \frac{3(\sqrt{5}-1)}{17\sqrt{5}-22}$$

and

(2.3.38) $$a_4\leq \frac{2(\sqrt{5}-1)}{17\sqrt{5}-22}$$

The estimate in (2.3.37) and (2.3.38) are sharp for the function

$$F(z) = z-\frac{12}{(17\sqrt{5}-22)}z^2-\frac{3(\sqrt{5}-1)}{17\sqrt{5}-22}z^3-\frac{2(\sqrt{5}-1)}{17\sqrt{5}-22}z^4-\frac{3}{17\sqrt{5}-22}z^5$$

2.4 In this section we find necessary condition satisfied by certain coefficients of functions belonging to the class $C_1(D)$.

Theorem 2.4.1 Let $f(z) = z-a_2z^2-a_{p+1}z^{p+1}$, $p \geq 2, (a_2 > 0, a_{p+1} \geq 0)$. Then $f \in C_1(D)$, if and only if,

$$(2.4.1) \quad a_{p+1} \leq \min \begin{cases} \dfrac{1-4a_2}{(p+1)^2} \\ \\ \dfrac{d_2}{p^2 d_{p+1}} a_2 \end{cases} .$$

Proof. The function $Df(z)$ is convex in Δ, if and only if,

$$g(z) = \frac{d_1 - Df(z)}{d_2 a_2} = z + \frac{d_{p+1} a_{p+1}}{d_2 a_2} z^p$$

is in K. We have $g(z)$ is in K, if and only if,

$$(2.4.2) \qquad \frac{d_{p+1} a_{p+1}}{d_2 a_2} \leq \frac{1}{p^2}$$

Therefore, it follows that $Df(z)$ is convex in Δ, if and only if, (2.4.2) holds. Again, f is in C, if and only if, (c.f. Section 1.2)

$$(2.4.3) \qquad 4a_2 + (p+1)^2 a_{p+1} \leq 1.$$

Now, $f \in C_1(D)$, if and only if, (2.4.2) and (2.4.3) hold and hence the result.

Remark. Theorem 2.4.1 gives a necessary and sufficient condition for a polynomial of the form $f(z) = z-a_2z^2-a_{p+1}z^{p+1}$, $p \geq 2$, to be in the class $C_1(D)$. We recall that an analogous result for $T_1(D)$ is given by Theorem 2.3.1.

Corollary 2.4.1 *Let* $f(z) = z-a_2z^2-a_{p+1}z^{p+1}$, $p \geq 2$, $(a_2 > 0, a_{p+1} \geq 0)$ *be in* $C_1(D)$. *Then* ,

$$(2.4.4) \qquad a_{p+1} \leq \frac{d_2}{4p^2d_{p+1}+(p+1)^2d_2}$$

The estimate in (2.4.4) *is sharp*.

Proof. Since $f \in C_1(D)$, from (2.4.1) , we get

$$(2.4.5) \qquad a_{p+1} \leq \frac{d_2}{p^2d_{p+1}}a_2 \leq \frac{d_2}{p^2d_{p+1}} \frac{1-(p+1)^2a_{p+1}}{4} .$$

This inequality yields

$$a_{p+1} \leq \frac{d_2}{4p^2d_{p+1}+(p+1)^2d_2}$$

which gives the coefficient bound in (2.4.4). To show that the estimate in (2.4.4) is sharp , we observe that equality in (2.4.5) holds everywhere , if and only if ,

$$a_2 = \frac{p^2d_{p+1}}{4p^2d_{p+1}+(p+1)^2d_2} \quad \text{and} \quad a_{p+1} = \frac{d_2}{4p^2d_{p+1}+(p+1)^2d_2}$$

Further , by (2.4.1), the function F given by

$$F(z) = z - \frac{p^2d_{p+1}}{4p^2d_{p+1}+(p+1)^2d_2} z^2 - \frac{d_2}{4p^2d_{p+1}+(p+1)^2d_2} z^{p+1}$$

is in $C_1(D)$. Thus , equality in (2.4.4) is attained for the function $F(z)$.

Remark. Corollary 2.4.1 gives a necessary condition satisfied by the (p+1)th coefficient of polynomials belonging to the class $C_1(D)$. Note that, an analogous result for $T_1(D)$ is given in Corollary 2.3.1.

Taking $p = 2$ and $d_n \equiv n$ in Corollary 2.4.1, we get the following coefficient estimate for cubic polynomials in the class C_1.

<u>Corollary</u> 2.4.2 <u>Let</u> $f(z) = z-a_2z^2-a_3z^3 (a_2 > 0, a_3 \geq 0)$ be in C_1. <u>Then</u>,

(2.4.6) $$a_3 \leq \frac{1}{33}$$

<u>The estimate in</u> (2.4.6) <u>is sharp</u>, <u>with equality for the function</u>

$$F(z) = z-\frac{2}{11}z^2 - \frac{1}{33}z^3$$

Putting $d_n \equiv 1$ in Corollary 2.4.1, we get

<u>Corollary</u> 2.4.3 <u>Let</u> $f(z) = z-a_2z^2-a_{p+1}z^{p+1}, (a_2 > 0, a_{p+1} \geq 0)$ <u>be in</u> C. <u>If</u> Lf <u>is convex in</u> Δ, <u>then</u>

(2.4.7) $$a_3 \leq \frac{1}{(5p^2+2p+1)}$$

The estimate in (2.4.7) is sharp for the function

$$F(z) = z - \frac{p^2}{5p^2+2p+1}z^2 - \frac{1}{5p^2+2p+1}\ z^{p+1}$$

<u>Remark</u>. For the class T, Corollary 2.3.2 gives a result analogous to the above corollary.

In the next theorem, we find a necessary condition satisfied by the third coefficient of any fourth degree polynomial belonging to the class $C_1(D)$.

<u>Theorem</u> 2.4.2 <u>Let</u> $f(z) = z-a_2z^2-a_3z^3-a_4z^4$ $(a_2>0, a_3>0, a_4>0)$ <u>be in</u> $C_1(D)$. <u>If</u> $d_1 \geq 1$, <u>then</u>

$$(2.4.8) \qquad a_3 \leq \frac{d_2\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}}{9d_3\left[\frac{2}{9}\left(1+\frac{4d_2}{9d_4}t_o\right)+\frac{d_2}{d_3}\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}\right]}$$

<u>where</u>, $t_o = (27-3\sqrt{6})/25$. <u>The estimate in</u> (2.4.8) <u>is sharp</u>.

<u>Proof</u>. We have $Df(z)$ is in $C_1(D)$, if and only if,

$$g(z) = \frac{d_1-Df(z)}{d_2a_2} = z+\frac{d_3a_3}{d_2a_2}z^2+\frac{d_4a_4}{d_2a_2}z^3$$

belong to K. Equivalently, by (1.2.12), $Df(z)$ is convex in Δ, if and only if,

$$(2.4.9) \qquad a_3 \leq \begin{cases} \dfrac{d_2a_2+9d_4a_4}{4d_3}, & 0 \leq a_4 \leq \dfrac{d_2}{15d_4}a_2 \\[2ex] \dfrac{2}{d_3}\sqrt{\dfrac{2d_4a_4(d_2a_2-9d_4a_4)d_2a_2}{3d_2a_2-25d_4a_4}}, & \dfrac{d_2}{15d_4}a_2 \leq a_4 \leq \dfrac{d_2}{9d_4}a_2. \end{cases}$$

Again, f is in C, if and only if,

$$(2.4.10) \qquad 9a_3 \leq 1-4a_2-16a_4.$$

Thus, from (2.4.9) and (2.4.10), $f \in C_1(D)$ if and only if, for $0 \leq a_4 \leq (d_2/15d_4)a_2$

$$(2.4.11) \qquad a_3 \leq \min\begin{cases} \dfrac{d_2a_2+9d_4a_4}{4d_3} \\[2ex] \dfrac{1-4a_2-16a_4}{9} \end{cases}$$

and, for $(d_2/15d_4)a_2 \leq a_4 \leq (d_2/9d_4)a_2$,

$$(2.4.12) \quad a_3 \leq \min \begin{cases} \dfrac{2}{d_3} \sqrt{\dfrac{2d_4a_4(d_2a_2-9d_4a_4)d_2a_2}{3d_2a_2-25d_4a_4}} \\ \\ \dfrac{1-4a_2-16a_4}{9} \end{cases}.$$

The right hand side of (2.4.11) will be maximized at a point for which

$$\frac{d_2a_2+9d_4a_4}{4d_3} = \frac{1-4a_2-16a_4}{9}.$$

That is, where

$$(2.4.13) \qquad (16d_3+9d_2)a_2 + (64d_3+81d_4)a_4 = 4d_3.$$

Set $a_4 = a_4(t) = \dfrac{d_2}{15d_4} ta_2$, $0 \leq t \leq 1$. Then, (2.4.13) yields

$$a_2 = a_2(t) = \frac{4d_3}{\left[(16d_3+9d_2)+\dfrac{d_2}{15d_4}(64d_3+81d_4)t\right]}.$$

Let,

$$h_1(t) = \frac{d_2a_2(t)+9d_4a_4(t)}{4d_3}$$

$$= \frac{d_2(1+\frac{3}{5}t)}{\left[(16d_3+9d_2)+\dfrac{d_2}{15d_4}(64d_3+81d_4)t\right]}.$$

Since, for $0 \leq t \leq 1$, $h_1'(t) = \dfrac{\dfrac{16d_2}{5}\left(3d_3-\dfrac{4d_2}{3d_4}\right)}{\left[(16d_3+9d_2)+\dfrac{d_2}{15d_4}(64d_3+81d_4)t\right]^2} > 0$,

$h_1(t)$ is an increasing function of t.

Therefore , $h_1(t) \leqq h_1(1) = (2d_2/5d_3)a_2(1)$. By (2.4.11), this gives for $o \leqq a_4 \leqq (d_2/15d_4)a_2$

$$(2.4.14) \qquad a_3 \leqq \frac{8d_2}{5[(16d_3+9d_2)+\frac{d_2}{15d_4}(64d_3+81d_4)]}$$

Similarly , the right side of (2.4.12) attains its maximum at a point where

$$(2.4.15) \qquad \frac{2}{d_3}\sqrt{\frac{2d_4a_4(d_2a_2-9d_4a_4)d_2a_2}{3d_2a_2-25d_4a_4}} = \frac{1-4a_2-16a_4}{9}$$

Setting $a_4=\tilde{a}_4(t)=\frac{d_2}{9d_4}ta_2$, $3/5 \leqq t \leqq 1$, we note that (2.4.15) gives

$$a_2=\tilde{a}_2(t) = \frac{1}{[18[\frac{2}{9}(1+\frac{4d_2}{9d_4}t)+\frac{d_2}{d_3}\sqrt{\frac{2t(1-t)}{27-25t}}]} , \quad 3/5 \leqq t \leqq 1 .$$

Define, $h_2(t)$ =

$$h_2(t) = \frac{1-4\tilde{a}_2(t)-16\tilde{a}_4(t)}{9} = \frac{1}{9} - \frac{4}{9}\tilde{a}_2(t)(1+\frac{4d_2}{9d_4}t)$$

$$h_2'(t) = \frac{(1-\frac{4d_2}{9d_4}t)(25t^2-54t+27)}{9 \quad 2t(1-t)(27-25t)^{3/2}[\frac{d_2}{d_3}\overline{\frac{2t(1-t)}{27-25t}}+\frac{2}{9}(1+\frac{4d_2}{9d_4}t)]} .$$

Now , $h_2'(t_o) = o$ for $t_o = (27-3\sqrt{6})/25$, and $h_2''(t_o) < o$. Since $3/5 \leqq t_o < 1$, $h_2(t)$, attains its maximum value at the point t_o , so that $h_2(t) \leqq h_2(t_o)$. By (2.4.12), we get

$$(2.4.16) \qquad a_3 \leqq \frac{d_2\sqrt{\frac{2t_o(1-t_o)}{27-25t_o}}}{9d_3[\frac{2}{9}(1+\frac{4d_2}{9d_4}t_o)+\frac{d_2}{d_3}\sqrt{\frac{2t_o(1-t_o)}{27-25t_o}}]}$$

where, $t_o = (27-3\sqrt{6})/25$. Since,

$$\frac{d_2\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}}{9d_3\left[\frac{2}{9}(1+\frac{4d_2}{9d_4}t_o)+\frac{d_2}{d_3}\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}\right]} > \frac{8d_2}{5[(16d_3+9d_2)+\frac{d_2}{15d_4}(64d_3+81d_4)]}$$

estimate in (2.4.8) holds for $o \leq a_4 \leq (d_2/9d_4)a_2$

To see that the coefficient bound in (2.4.8) is sharp consider the function F, given by

$$F(z) = z-a_2z^2-a_3z^3-a_4z^4 ,$$

where,

$$a_2 = \frac{1}{18\left[\frac{2}{9}(1+\frac{4d_2}{9d_4}t_o)+\frac{d_2}{d_3}\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}\right]}$$

$$a_3 = \frac{d_2\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}}{9d_3\left[\frac{2}{9}(1+\frac{4d_2}{9d_4}t_o)+\frac{d_2}{d_3}\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}\right]}, \quad a_4 = \frac{d_2}{9d_4}t_o a_2$$

Since, $4a_2+9a_3+16a_4 = 1$, $F(z)$ belongs to C. Further, by (1.2.12), $DF(z)$ is convex in Δ. Thus, $F(z)$ is in $C_1(D)$ and the bound in (2.4.8) is attained by the function $F(z)$.

Taking $d_n = n$, $n = 1, 2, \ldots,$ in Theorem 2.4.2, we get the following coefficient bound for fourth degree polynomials belonging to the class C_1.

Corollary 2.4.4 Let $f(z) = z-a_2z^2-a_3z^3-a_4z^4 (a_2>0, a_3>0, a_4>0)$ be in C_1. Then,

$$(2.4.17)\qquad a_3 \le \frac{2\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}}{27\left[\frac{2}{9}(1+\frac{2}{9}t_o)+\frac{2}{3}\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}\right]}$$

where, $t_o = (27-3\sqrt{6})/25$. The estimate in (2.4.17) is sharp.

With $d_n = 1$, $n = 1, 2, \ldots$ Theorem 2.4.2, gives the following coefficient estimate for the shift operator.

Corollary 2.4.5 Let $f(z) = z-a_2z^2-a_3z^3-a_4z^4$, $(a_2>0, a_3>0, a_4>0)$ be in C. If the shift operator Lf is convex in Δ, then

$$(2.4.18)\qquad a_3 \le \frac{\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}}{\left[2(1+\frac{4}{9}t_o)+9\sqrt{\dfrac{2t_o(1-t_o)}{27-25t_o}}\right]}$$

The estimate in (2.4.18) is sharp.

2.5 In this section, we find coefficient bounds for functions in the class $T_1(D)$ which are not necessarily polynomials. We observe that, since $T_1(D)$ is a sub class of T, the inequality, $a_2 \le 1/2$, continues to holds even for functions belonging to the class $T_1(D)$. That this inequality is sharp can be seen by considering the function $F(z) = z-z^2/2 \in T_1(D)$. We now find a bound on the third coefficient of a function in $T_1(D)$ in the following theorem:

Theorem 2.5.1 *Suppose* $f(z) = z - \sum_{n=0}^{\infty} a_n z^n$ $(a_n \geq 0)$, *belongs to* $T_1(D)$ *and* $\beta_o = \sup_{n \geq 2}\{\frac{d_n}{n}\}$. *If* $\beta_o < \infty$ and $o < a_2 \leq \frac{\beta_o}{d_2}$, *then*

$$(2.5.1) \qquad a_3 < \frac{\beta_o}{2d_3} .$$

Proof. Since $Df(z)$ is univalent in Δ, the function

$$(2.5.2) \qquad g(z) = \frac{d_1 - Df(z)}{d_2 a_2}$$

$$= z + \sum_{n=2}^{\infty} \frac{d_{n+1} a_{n+1}}{d_2 a_2} z^n$$

is in S. Again, from the hypothesis $d_n \leq \beta n$, $n = 2, 3, \cdots$ and since $f \in T$, we have (c.f. Section 1.2), $\sum_{n=2}^{\infty} n a_n \leq 1$. Thus,

$$(2.5.3) \qquad \sum_{n=3}^{\infty} d_n a_n \leq \beta_o - d_2 a_2$$

Using (2.5.3), we get

$$|g(z)| \leq |g(1)| \leq 1 + \sum_{n=2}^{\infty} \frac{d_{n+1} a_{n+1}}{d_2 a_2}$$

$$\leq \frac{\beta_o}{d_2 a_2} .$$

Since $g \in S$ and $|g(z)| \leq \frac{\beta_o}{d_2 a_2}$, $\frac{\beta_o}{d_2 a_2} > 1$, by (1.2.2), we have

$$(2.5.4) \qquad a_3 \leq \frac{d_2}{d_3} a_2 \left(1 - \frac{d_2 a_2}{\beta_o}\right) .$$

The right side of the inequality in (2.5.4) attains its maximum at $a_2 = \beta_o/2d_2$. So by putting $a_2 = \beta_o/2d_2$ in the right side of this inequality we obtain

$$a_3 \leq \frac{\beta_o}{2d_3} .$$

To show that strict inequality holds, we assume that $a_3 = \beta_o/2d_3$. So that (2.4.4) is equivalent to

$$1 \leqq \frac{4d_2}{\beta_o}a_2(1 - \frac{d_2}{\beta_o}a_2) = 1-(1 - \frac{2d_2}{\beta_o}a_2)^2$$

from which we deduce that $a_2 = \beta_o/2d_2$. Thus, $a_2 = \beta_o/2d_2$, $a_3 = \beta_o/2d_3$, and hence

$$\sum_{n=4}^{\infty} d_n a_n \leqq \beta_o - d_2a_2 - d_3a_3 = 0 .$$

Since $d_n \neq o$ for any n, this implies that $a_n = o$ for $n = 4, 5, \ldots$ Therefore, $f(z)$ is a cubic polynomial. But, by Corollary 2.3.1, $a_3 \leqq d_2/(4d_3+3d_2)$ and the estimate is sharp. Since $d_2/(4d_3+3d_2) < \beta_o/2d_3$, we have

$$a_3 < \frac{\beta_o}{2d_3} .$$

This proves the inequality (2.5.1).

<u>Remark</u>. Taking $d_n = n$, $n = 1, 2, \ldots$, in Theorem 2.5.1, we get $\beta_o = 1$ and hence $a_3 < 1/6$, a result due to Silverman [94].

Putting $d_n = 1$, $n = 1, 2, \ldots$, in Theorem 2.5.1, we get

<u>Corollary</u> 2.5.1 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geqq o)$ <u>be in</u> T. <u>If</u> Lf <u>is univalent in</u> Δ, <u>then</u>

$$a_3 < \frac{1}{4} . \tag{2.5.5}$$

In the next theorem upper bounds for the fourth coefficient of functions belonging to the class $T_1(D)$ have been determined.

<u>Theorem</u> 2.5.2 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, <u>be in</u> $T_1(D)$ <u>and</u> $\beta_o = \sup_{n \geq 2} \{\frac{d_n}{n}\}$. <u>If</u> $\beta_o < \infty$ <u>and</u> $o < a_2 \leq \frac{\beta_o}{d_2}$, <u>then</u>

$$
(2.5.6) \qquad a_4 \leq \begin{cases} \dfrac{\beta_o(3e-1)(e-1)}{d_4 e^3}, & o < a_2 \leq \dfrac{\beta_o}{d_2 e} \\[2ex] \dfrac{2\beta_o}{3\sqrt{3}\, d_4}, & \dfrac{\beta_o}{d_2 e} < a_2 \leq \dfrac{\beta}{d_2}. \end{cases}
$$

<u>Proof</u>. Let $g(z)$ be defined by (2.5.2). Then $g(z)$ belongs to S and $|g(z)| < M$ where $M = \frac{\beta_o}{d_2 a_2}$. Therefore, by (1.2.3),

$$
(2.5.7) \qquad a_4 \leq \begin{cases} a_2\left[2\left(\sigma - \dfrac{d_2 a_2}{\beta_o}\right)^2 + \left(1 - \dfrac{(d_2 a_2)^2}{\beta_o^2}\right)\right], & o < a_2 \leq \dfrac{\beta_o}{d_2 e} \\[2ex] a_2\left(1 - \dfrac{(d_2 a_2)^2}{\beta_o^2}\right), & \dfrac{\beta_o}{d_2 e} < a_2 \leq \dfrac{\beta_o}{d_2}. \end{cases}
$$

where, σ is the real root of $Mx \log x = -1$.

We now maximise the right hand side of (2.5.7) for $o < a_2 \leq \frac{\beta_o}{d_2}$.

Let

$$h_\sigma(x) = x\left[2(\sigma - \alpha x)^2 + (1 - \alpha^2 x^2)\right], \quad \alpha = \frac{d_2}{\beta_o}.$$

We observe that, $x \in (o, \frac{1}{\alpha e}]$, if and only if, $\sigma \in [\frac{1}{e}, 1)$. Since

$$\frac{d}{d\sigma} h_\sigma(x) = 4x(\sigma - \alpha x) \geq o$$

$h_\sigma(x)$ is an increasing function of σ, $\frac{1}{e} \leq \sigma < 1$. Therefore,

$$\max_{\frac{1}{e} \leq \sigma < 1} h_\sigma(x) = \lim_{\sigma \to 1} h_\sigma(x) = x(\alpha^2 x^2 - 4\alpha x + 3) \equiv p(x) \text{ (say)}.$$

Now, for $o < x \leq \frac{1}{\alpha e}$

$$p'(x) = 3\alpha^2x^2 - 8\alpha x + 3 > \frac{3e-8}{e} > o.$$

Therefore, $p(x)$ is an increasing function of x., so that

$$\max_{x\epsilon(o,\frac{1}{\alpha e}]} p(x) = \frac{(3e-1)(e-1)}{\alpha e^3}$$

From this, we conclude that, for $o < a_2 \leq \frac{\beta_o}{d_2 e}$,

(2.5.8) $$a_4 \leq \frac{\beta_o(3e-1)(e-1)}{d_4 e^3}.$$

Similarly for $\frac{\beta_o}{d_2 e} < a_2 \leq \frac{\beta_o}{d_2}$, the right side of the inequality in (2.5.7) is maximum at $a_2 = \beta_o/d_2\sqrt{3}$. Therefore for $\beta_o/d_2 e < a_2 \leq \beta_o/d_2$, putting $a_2 = \beta_o/d_2\sqrt{3}$ in the inequality in (2.5.7), we get

(2.5.9) $$a_4 \leq \frac{2\beta_o}{3\sqrt{3}d_4}.$$

With $d_n = n$, $n = 1, 2, \ldots,$ in Theorem 2.5.2, we get

<u>Corollary</u> 2.5.2 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> T_1. <u>Then</u>,

(2.5.10) $$a_4 \leq \begin{cases} \dfrac{(3e-1)(e-1)}{4e^3}, & o < a_2 \leq \dfrac{1}{2e} \\[2ex] \dfrac{1}{6\sqrt{3}}, & \dfrac{1}{2e} < a_2 \leq \dfrac{1}{2}. \end{cases}$$

Taking $d_n = 1$, $n = 1, 2, \ldots$, in Theorem 2.5.2, we have the following result:

<u>Corollary</u> 2.5.3 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> T . <u>If</u> Lf <u>is univalent in</u> Δ , <u>then</u>

$$(2.5.11) \qquad a_4 \leq \begin{cases} \dfrac{(3e-1)(e-1)}{2e^3}, & o < a_2 \leq \dfrac{1}{2e} \\[2ex] \dfrac{1}{3\sqrt{3}}, & \dfrac{1}{2e} < a_2 \leq \dfrac{1}{2} \end{cases}$$

Combining Corollary 2.3.3 and Corollary 2.5.1 , we have

<u>Corollary</u> 2.5.4 <u>Let</u> $\gamma_o = \sup \{a_3 : f \text{ and } Lf \text{ are univalent in } \Delta\}$. <u>Then</u> ,

$$(2.5.12) \qquad \frac{1}{(3+2\sqrt{3})} \leq \gamma_o < \frac{1}{4} .$$

2.6 We now find sufficient conditions on the coefficients of functions in T which force their Gelfond-Leontev derivatives of higher order to be univalent in Δ . We also find sufficient conditions which force the higher order Gelfond-Leontev derivatives of a function in C to be convex in Δ . First we have the following definitions .

<u>Definition</u> 2.6.1 A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq o$, is said to be in the class $T_m(D)$ if f and its first m Gelfond-Leontev derivatives are analytic and univalent in Δ . If $f \in T_m(D)$ for all m , then f is said to be in the class $T_\infty(D)$.

<u>Definition</u> 2.6.2 A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, is said to be in the class $C_m(D)$ if f and its first m Gelfond-Leontev derivatives are analytic and convex in Δ. If $f \in C_m(D)$ for all m, then f is said to be in the class $C_\infty(D)$.

We prove

<u>Theorem</u> 2.6.1 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> T <u>with</u> $\prod_{n=2}^{m+1} a_n \neq 0$. <u>If</u>

$$\sum_{n=k+2}^{\infty} (n-k)(d_{n-k+1} \cdots d_n) a_n \leq (d_{k+1} \cdots d_2) a_{k+1} \tag{2.6.1}$$

<u>for</u> $k = 1, 2, \ldots, m$, <u>then</u> $f \in T_m(D)$.

<u>Proof</u>. The case $m = 1$ was proved in Theorem 2.3.2. So, let $k > 1$. The function $D^k f(z)$ is univalent in Δ, if and only if,

$$g_k(z) = -\frac{D^k f(z) + (d_k \cdots d_1) a_k}{(d_{k+1} \cdots d_2) a_{k+1}} \tag{2.6.2}$$

$$= z + \sum_{n=2}^{\infty} \frac{d_{n+k} \cdots d_{n+1}}{d_{k+1} \cdots d_2} \frac{a_{n+k}}{a_{k+1}} z^n$$

$$= z + \sum_{n=2}^{\infty} b_n z^n .$$

is in the class S. Since $\sum_{n=2}^{\infty} n b_n = \sum_{n=2}^{\infty} n \frac{d_{n+k} \cdots d_{n+1}}{d_{k+1} \cdots d_2} \frac{a_{n+k}}{a_{k+1}} \leq 1$ is a sufficient condition for $g_k(z)$ to be in S, $D^k f(z)$ is univalent in Δ. Thus, $f \in T_m(D)$.

<u>Corollary</u> 2.6.1 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> T <u>with</u> $a_n \neq 0$. <u>If</u> (2.6.1) <u>holds for all</u> $k = 1, 2, \ldots,$ <u>then</u> $f \in T_\infty(D)$.

Taking $d_n = 1$, $n = 1, 2, \ldots,$ in Theorem 2.6.1 , we get the following sufficient condition for the shift operator.

<u>Corollary</u> 2.6.2 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $\prod_{n=2}^{m+1} a_n \neq 0$, <u>be in</u> T. <u>If</u>

$$\sum_{n=k+2}^{\infty} (n-k)\, a_n \leq a_{k+1} \tag{2.6.3}$$

<u>for</u> $k = 1, 2, \ldots, m$, <u>then</u> $L^k f$ <u>is univalent in</u> Δ <u>for</u> $k = 1, 2, \ldots, m$.

We observe in our next theorem that there is no extremal function for the second coefficient of functions belonging to the class $T_\infty(D)$.

<u>Theorem</u> 2.6.2 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> $T_\infty(D)$. <u>Then</u> , $a_2 < \frac{1}{2}$. <u>The result is sharp in the sense that the value</u> 1/2 <u>can not be replaced by any smaller constant and further, the value</u> 1/2 <u>is not attained by any function in</u> $T_\infty(D)$.

<u>Proof</u>. First we show that for given $\eta > 0$, there exists a function in $T_\infty(D)$ whose second coefficient is $(1-\eta)/2$. This will prove that the value 1/2 can not be replaced by any smaller constant . To this end, let for any η , $0 < \eta \leq \epsilon_1 = d_2/(d_2+3)$,

$$F_\eta(z) = z - \frac{(1-\eta)}{2} z^2 - \sum_{n=3}^{\infty} \frac{1}{2^{n-3} d_n \cdots d_1} z^n$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ of non-negative numbers are chosen such that

$$\text{(i)}\ d_1 d_2 \geq 2 \quad \text{(ii)}\ \frac{n}{d_1 d_2 \cdots d_n} \leq \frac{1}{2}, \quad n = 2, 3, \ldots . \tag{2.6.4}$$

(e.g., $d_n = n^\alpha$, $\alpha \geq 1$ or e^n etc).

Since ,

$$\frac{2(1-\eta)}{2} + \eta \sum_{n=3}^{\infty} \frac{n}{2^{n-3} d_n \cdots d_1} < 1$$

by (1.2.26) , the function $F_\eta(z)$, is in T .

Next we show that $F_\eta \in T_\infty(D)$. In view of Corollary 2.6.1 , it is sufficient to show that its coefficient satisfy (2.6.1) , for every k = 1, 2, ... Due to our choice of the sequence $\{d_n\}_{n=1}^{\infty}$ and η , we have

$$\eta \sum_{n=3}^{\infty} \frac{(n-1)d_n}{2^{n-3} d_n \cdots d_1} = \frac{2\eta}{d_1 d_2} + \eta \sum_{n=3}^{\infty} \frac{n}{2^{n-2} d_n \cdots d_1}$$

$$\leq \frac{3\eta}{2} < \frac{1-\eta}{2} d_2$$

so that (2.6.1) is satisfied for k = 1 . Further , for k >1 due to our choices of $\{d_n\}_{n=1}^{\infty}$ and η

$$\sum_{n=k+2}^{\infty} (n-k) d_{n-k+1} \cdots d_n a_n = \eta \sum_{n=k+2}^{\infty} (n-k) \frac{d_{n-k+1} \cdots d_n}{2^{n-3} d_n \cdots d_1}$$

$$< \frac{3\eta}{2^k} < (d_{k+1} \cdots d_1) a_{k+1}$$

which shows that (2.6.1) is satisfied for k > 1. Thus , for any η , $0 < \eta < d_2/(d_2+3)$. $F_\eta \in T_\infty(D)$ and has second coefficient $(1-\eta)/2$.

Since the function $f(z) = z - z^2/2$ is not in $T_\infty(D)$ and is the unique extremal function in $T(\supset T_\infty(D))$ for the second coefficient [94] , there is no extremal function in the class $T_\infty(D)$.

<u>Remarks</u> 1. The family $T_\infty(D)$ is not compact. To see this, we consider the sequence $\{F_k(z)\}_{k=1}^{\infty}$ of functions defined by

$$F_k(z) = z - \frac{(1-\frac{1}{k})}{2} z^2 - \frac{1}{k} \sum_{n=3}^{\infty} \frac{1}{2^{n-3}(d_n \cdots d_1)} z^n$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ is chosen as in (2.6.4). Then, for $k \geq 3$, $F_k \in T_\infty(D)$. But, clearly, the sequence $\{F_k(z)\}_{k=1}^{\infty}$ converges uniformly on compact subsets of Δ to the function $F(z) = z-z^2/2$ which is not in the family $T_\infty(D)$.

2. Putting $d_n = n$, $n = 1, 2, \ldots,$ in Theorem 2.6.1, Corollary 2.6.1, we get the corresponding results of Silverman [94].

Next, we determine sufficient conditions on the coefficients which force a function in the class C to belong to the class $C_m(D)$, i.e. the first m Gelfond-Leontev derivatives of the function are also forced to be convex is Δ.

<u>Theorem</u> 2.6.3 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> C <u>with</u> $\prod_{n=2}^{m+1} a_n \neq 0$. <u>If</u>

$$\sum_{n=k+2}^{\infty} (n-k)^2 \, d_{n-k+1} \cdots d_n a_n \leq (d_{k+1} \cdots d_2) a_{k+1} \tag{2.6.5}$$

<u>for</u> $k = 1, 2, \ldots, m$, <u>then</u> $f \in C_m(D)$.

<u>Proof</u>. The case m = 1 was proved in Theorem 2.2.5. So, let $k > 1$. The function $D^k f(z)$ is convex in Δ, if and only if, the function $g_k(z)$ defined by (2.6.2) is convex in Δ. Since,

$$\sum_{n=2}^{\infty} n^2 b_n = \sum_{n=2}^{\infty} n^2 \frac{d_{n+k} \cdots d_{n+1}}{d_{k+1} \cdots d_2} \frac{a_{n+k}}{a_{k+1}} \leq 1$$

is a sufficient condition for the function $g_k(z)$ to be convex in Δ and by (2.6.5), it clearly holds, $D^k f(z)$ is convex in Δ. Thus, $f \in C_m(D)$.

<u>Corollary</u> 2.6.3 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> C <u>with</u> $a_n \neq 0$. <u>If</u> (2.6.5) <u>holds for every</u> k, <u>then</u> $f \in C_\infty(D)$.

Putting $d_n = n$, $n = 1, 2, \ldots,$ in Theorem 2.6.3, we get the following sufficient condition for functions to be in the class C_m.

<u>Corollary</u> 2.6.4 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> C <u>with</u> $\prod_{n=2}^{m+1} a_n \neq 0$. <u>If</u>

$$(2.6.6) \qquad \sum_{n=k+2}^{\infty} (n-k)^2 (n-k+1) \ldots n \, a_n \leq (k+1)! \, a_{k+1}$$

<u>for</u> $k = 1, 2, \ldots, m$ <u>then</u> $f \in C_m$.

With $d_n = 1$, $n = 1, 2, \ldots$ in Theorem 2.6.3, we have

<u>Corollary</u> 2.6.5 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> C <u>with</u> $\prod_{n=2}^{m+1} a_n \neq 0$. <u>If</u>

$$(2.6.7) \qquad \sum_{n=k+2}^{\infty} (n-k)^2 a_n \leq a_{k+1}$$

<u>for</u> $k = 1, 2, \ldots, m$, <u>then</u> $Lf, L^2 f, \ldots, L^m f$ <u>are convex in</u> Δ.

The following theorem shows that there is no extremal function for the second coefficient of functions in the class $C_\infty(D)$ as well.

<u>Theorem</u> 2.6.4 <u>Let</u> $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> $C_\infty(D)$. <u>Then</u>, $a_2 < 1/4$. <u>The result is sharp in the sense that the constant</u> 1/4 <u>can not be replaced by any smaller constant and further, the value</u> 1/4 <u>is not attained by any function in</u> $C_\infty(D)$.

Proof. First, we show that for an arbitrary $\eta > o$, there exists a function in $C_\infty(D)$ whose second coefficient is $(1-\eta)/4$. To see this, let $o < \eta \leq d_2d_3/(d_2d_3+6)$ and set

$$F_\eta(z) = z - \frac{(1-\eta)}{4}z^2 - \eta \sum_{n=3}^{\infty} \frac{1}{2^{n-3}(d_n\cdots d_1)^2} z^n$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ of non-decreasing positive numbers are chosen such that $d_1 = 1$ and

(2.6.8) (i) $d_2 \geq 2$ (ii) $n/(d_n\cdots d_2) \leq 1/\sqrt{2}$ for $n = 3,4,\cdots$

Since,

$$4\left(\frac{1-\eta}{4}\right) + \eta \sum_{n=3}^{\infty} \frac{n^2}{2^{n-3}(d_n\cdots d_2)^3}$$

$$< (1-\eta)+\eta \sum_{n=3}^{\infty} \frac{1}{2^{n-3}}$$

$$= 1.$$

the function $F_\eta \varepsilon C$, by (1.2.28). Next, we show that $F_\eta \varepsilon C_\infty(D)$. In view of Corollary 2.6.3, it is enough to show that its coefficients satisfy (2.6.5) for every k. Due to our choice of $\{d_n\}_{n=1}^{\infty}$ and η,

$$\sum_{n=3}^{\infty} (n-1)^2 d_n a_n = \eta \sum_{n=3}^{\infty} \frac{(n-1)^2 d_n}{2^{n-3}(d_n\cdots d_2)^2}$$

$$< \frac{4\eta}{d_2^2 d_3} + \frac{\eta}{d_3} \sum_{n=4}^{\infty} \left(\frac{n-1}{d_{n-1}\cdots d_2}\right)^2 \frac{1}{2^{n-3}}$$

$$\leq \frac{\eta}{d_3} + \frac{\eta}{d_3} \sum_{n=4}^{\infty} \frac{1}{2^{n-2}} < d_2\left(\frac{1-\eta}{4}\right).$$

Thus, (2.6.5) is true for k = 1.

Again , for $k > 1$ and due to choice of $\{d_n\}_{n=1}^{\infty}$ and η

$$\sum_{n=k+2}^{\infty} (n-k)^2 d_{n-k+1}\cdots d_n a_n = \eta \sum_{n=k+2}^{\infty} \frac{(n-k)^2 d_{n-k+1}\cdots d_n}{(d_n\cdot d_2)^2 2^{n-3}}$$

$$\leq \frac{\eta}{2^{k-1}(d_{k+1}\cdots d_2)} + \eta \sum_{n=3}^{\infty} \left(\frac{n}{d_n\cdots d_2}\right) \frac{1}{d_{k+1}\cdots d_2}$$

$$< \frac{3}{4} \frac{\eta}{2^{k-2}(d_{k+1}\cdots d_2)} < (d_{k+1}\cdots d_2)\, a_{k+1} .$$

Thus , (2.6.5) is satisfied for $k = 1, 2, 3, \ldots$. Hence for any η , $0 < \eta \leq d_2 d_3/(d_2 d_3+6)$, $F_\eta \in C_\infty(D)$ and the second coefficient is $(1-\eta)/4$.

Further , since the function $F(z) = z - z^2/4$ is not in $C_\infty(D)$ and is the only extremal function in $C(\supset C_\infty(D))$ for the second coefficient there is no extremal function in $C_\infty(D)$.

Remark. The family $C_\infty(D)$ is not compact : For this , we consider the sequence $\{F_k(z)\}_{k=1}^{\infty}$ of functions defined by

$$F_k(z) = z - \frac{(1-\frac{1}{k})}{4} z^2 - \frac{1}{k} \sum_{n=3}^{\infty} \frac{1}{2^{n-3}(d_n\cdots d_2)^2} z^n$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ satisfies (2.6.8) . Then for $k \geq 3$, by Theorem 2.6.4 $F_k \in C_\infty(D)$. But , $\{F_k(z)\}_{k=1}^{\infty}$ converges uniformly on compact subsets of Δ to the function $F(z) = z-z^2/4$ which is not in the class $C_\infty(D)$.

CHAPTER III

UNIVALENT FUNCTIONS WITH SOME UNIVALENT GELFOND-LEONTEV DERIVATIVES

3.1 Let H_R, $0 < R \leqq \infty$, denote the class of functions of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, that are analytic in the disc $\{z : |z| < R\}$. As observed in sections 1.5 and 1.6, functions in the class H_R having some of the ordinary derivatives univalent in Δ are widely studied. In the present chapter, we extend this study further by considering the implications of the univalence of some of the Gelfond-Leontev derivatives of functions f belonging to H_R, instead of considering just the ordinary derivatives of f.

For a strictly increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers and a strictly increasing sequence $\{d_n\}_{n=1}^{\infty}$ of positive numbers, set

$$D^{n_p} f(z) = \sum_{k=0}^{\infty} d_{n_p+k} \cdots d_{k+1}\, a_{n_p+k}\, z^k \tag{3.1.1}$$

We consider the following two subclasses $E(n_p, R) \equiv E(\{n_p\}, R)$ and $E_c(n_p, R) \equiv E_c(\{n_p\}, R)$ of the set of analytic and univalent functions in the unit disc Δ:

$$E(n_p, R) = \{f \in H_R : D^{n_p} f,\ p = 1, 2, \ldots, \text{is analytic and univalent in } \Delta\}$$

$$E_c(n_p, R) = \{f \in H_R : D^{n_p} f,\ p = 1, 2, \ldots, \text{is analytic, univalent and convex in } \Delta\}$$

Clearly , $E_c(n_p;R) \subset E(n_p;R)$. Further , the classes $E(n_p;\infty)$ and $E_c(n_p;\infty)$ have obvious meaning .

If $f \in E(p;R)$, i.e. if $n_p \equiv p$, then $\rho_p \equiv 1$ and further, if $d_p \to \infty$, then (1.5.13) implies that f is an entire function. Likewise , if $f \in E(p;\infty)$ then $\rho_p \equiv 1$ and if $(d_p^{\Lambda}/p) \to \infty$ as $p \to \infty$, $o < \Lambda < \infty$, then (1.5.20) gives that $\tau = o$, where τ is the type of f. However , if infinitely many ρ_p's are zero , then (1.5.13) and (1.5.20) do not give any non-trivial information about the radius of convergence R of f and the type τ of an entire function . To this end , instead of assuming f to be in the class $E(p;R)$ or $E(p;\infty)$, we assume f to be in the class $E(n_p;R)$ or $E(n_p;\infty)$ and partially remedy the situation. A similar problem is also studied for the classes $E_c(n_p;R)$ and $E_c(n_p;\infty)$.

In Section 3.2 , we find relations between the sequence $\{d_{n_p}\}_{p=1}^{\infty}$ of positive numbers and the radius of convergence R of functions belonging to the class $E(n_p;R)$ or $E_c(n_p;R)$. Some sufficient conditions on the sequence $\{n_p\}_{p=1}^{\infty}$, (the exponents of the Gelfond-Leontev derivative $D^{n_p}f$ which are analytic and univalent(or convex univalent)) that force the functions in $E(n_p;R)$ or $E_c(n_p;R)$ to be entire are also found in this section . In section 3.3 , by restricting the growth of the sequence $\{n_p-n_{p-1}\}_{p=2}^{\infty}$ by slowly oscillating functions , the order of an entire function in $E(n_p;\infty)$ is estimated .

Section 3.4 is devoted to finding relations between the sequence $\{d_{n_p}\}_{p=1}^{\infty}$ with the order and type of an entire function belonging to the class $E(n_p, R)$ or $E_c(n_p, R)$. Furthermore, we find sufficient conditions on the sequences $\{n_p\}_{p=1}^{\infty}$ and $\{d_{n_p}\}_{p=1}^{\infty}$ for f to have minimal type. Some of the results obtained in this chapter include the work of Shah and Trimble [83] and Campbell [19].

For the convenience in the use of notations, throughout in the sequel we write $a(n_p)$ for a_{n_p}, $d(n_p)$ for d_{n_p}.

3.2 The relations between sequence $\{d(n_p)\}_{p=1}^{\infty}$ of positive numbers and the radius of convergence R of functions belonging to the classes $E(n_p, R)$ and $E_c(n_p, R)$ (c.f. Section 3.1) are found in this section. These relations are then used to obtain sufficient conditions on the sequence $\{n_p\}$ that force a function in $E(n_p, R)$ or $E_c(n_p, R)$ to be entire.

<u>Theorem</u> 3.2.1 <u>Let</u> f <u>be in</u> $E(n_p, R)$ <u>and the sequence</u> $\{d_n\}_{n=1}^{\infty}$ <u>in</u> (3.1.1) <u>satisfies</u>

$$(3.2.1) \qquad \frac{d(n_p+k)}{d(k)} \geq \frac{k}{k-1}\ ; \ k = 2, 3, \ldots\ ; \ p \geq p_0 \text{ for some } p_0 .$$

<u>Then</u>,

$$(3.2.2) \qquad \liminf_{p \to \infty} \{d(n_1) d(n_2) \ldots d(n_p)\}^{1/n_{p+1}} \leq R \limsup_{p \to \infty} (2d_2)^{p/n_p}$$

$$\leq 2d_2 R.$$

Proof. Since $f(z) = \sum_{n=0}^{\infty} a_n z^n \varepsilon E(n_p, R)$, we have that $D^{n_p}f$ is analytic and univalent in the unit disc Δ. This implies (c.f. Section 1.2) that for $p = 1, 2, \ldots,$ and $k = 2, 3, \ldots$

$$(3.2.3) \qquad |a(n_p+k)| \leq \frac{kd(n_p+1)\ldots d(2)|a(n_p+1)|}{d(n_p+k)\ldots d(k+1)}$$

Due to (3.2.1), the function $k/d(n_p+k)\ldots d(k+1)$ decreases with k for some $p \geq p_o$. Therefore, for $k = 2, 3, \ldots,$ and $p \geq p_o$, (3.2.3) becomes

$$(3.2.4) \qquad |a(n_p+k)| \leq \frac{2d(n_p+1)\ldots d(2)|a(n_p+1)|}{d(n_p+2)\ldots d(3)}$$

$$< \frac{2d(2)|a(n_p+1)|}{d(n_p)}$$

An inductive argument on p in (3.2.4) yields for $p \geq p_o$,

$$|a(n_p+1)| \leq \frac{(2d(2))^{p-p_o}|a(n_1+1)|d(n_1+1)\ldots d(2)}{d(n_{p_o})\ldots d(n_{p-1})}$$

If $2 \leq k \leq n_{p+1}-n_p+1$, then by using the above inequality in (3.2.4), we get for $p \geq p_o$

$$(3.2.5) \qquad |a(n_p+k)| < \frac{(2d(2))^{p-p_o+1}|a(n_1+1)|d(n_1+1)\ldots d(2)}{d(n_{p_o})\ldots d(n_p)}$$

$$= \frac{A_1(2d(2))^p}{d(n_1)d(n_2)\ldots d(n_p)}$$

where, A_1 is a constant. From the inequality in (3.2.5),

we deduce that

Proof. Since $D^{n_p}f$ is analytic univalent and convex in Δ, it follows from (1.2.9) that

$$(3.2.8) \qquad |a(n_p+k)| \leq \frac{d(n_p+1)\ldots d(2)|a(n_p+1)|}{d(n_p+k)\ldots d(k+1)}$$

for $p = 1, 2, \ldots$ and $k = 2, 3, \ldots$ But, the function $1/d(n_p+k)\ldots d(k+1)$ decreases with k for every p, so that (3.2.8) gives for $p = 1, 2, \ldots$, and $k = 2, 3, \ldots$

$$|a(n_p+k)| \leq \frac{d(n_p+1)\ldots d(2)|a(n_p+1)|}{d(n_p+2)\ldots d(k+1)}$$

$$< \frac{d(2)|a(n_p+1)|}{d(n_p)} .$$

Now, by following the same lines of proof as in Theorem 3.2.1, we get

$$\frac{1}{R} \leq \frac{\limsup_{p\to\infty} d_2^{p/n_p}}{\liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_p)\}^{1/n_{p+1}}} .$$

This proves the theorem.

Corollary 3.2.2 Let f be in $E_c(n_p, R)$. If $(n_{p+1}-n_p) = o(\log d(n_p))$ then f is entire.

Proof. Since $n_{p+1}-n_p = o(\log d(n_p))$, Corollary 3.2.2 follows from Theorem 3.2.2 and the inequality in (3.2.6).

<u>Remarks</u>. 1. We note that (3.2.1) holds for $d_n \equiv n^a$, $a > o$. In general, (3.2.1) is satisfied if $d(k)/(d(k)-1) \geq k/(k-1)$, $k = 2, 3, \ldots$. Using the later inequality a number of examples of admissible sequences $\{d_n\}_{n=1}^{\infty}$ for Theorem 3.2.1 could be easily generated.

2. Let $\lim_{n \to \infty} d_n^{1/n} = b$, $o < b < \infty$. Then, the inequality in (3.2.2) and (3.2.7) can be written as

$$(3.2.9) \quad \liminf_{p \to \infty} \{d(n_1)d(n_2)\ldots d(n_p)\}^{1/n_p} \leq bR \limsup_{p \to \infty} (2d_2)^{p/n_p} \leq b(2d_2R).$$

and

$$(3.2.10) \quad \liminf_{p \to \infty} \{d(n_1)d(n_2)\ldots d(n_p)\}^{1/n_p} \leq bR \limsup_{p \to \infty} d_2^{p/n_p}. \leq bd_2R$$

3. For $d_n \equiv n$, the inequality in (3.2.9) gives a result of Shah and Trimble [83]. While in this particular case the inequality in (3.2.10) gives a result of Campbell [19].

If $f \in E(n_p, R)$, d_n's in (3.1.1) satisfy (3.2.1) and $\limsup_{p \to \infty}(n_{p+1}-n_p) < \infty$ then it follows from corollary 3.2.1 that f is entire. Similarly, if $f \in E_c(n_p, R)$ and $\limsup_{p \to \infty}(n_{p+1}-n_p) < \infty$, Corollary 3.2.2 gives that f is entire. The following theorems are in the other direction:

<u>Theorem</u> 3.2.3 <u>Let</u> f <u>be in</u> $E(n_p, R)$ <u>and the sequence</u> d_n's <u>in</u> (3.1.1) <u>satisfies</u> (3.2.1). <u>If there is a positive integer</u> $M > 1$ <u>such that</u> $\liminf_{p \to \infty} (n_{p+1}-n_p) \geq M$, <u>then</u>

$$(3.2.11) \quad \liminf_{p \to \infty} \{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{(M-1)/n_{p+1}} \leq R.$$

<u>Proof</u>. Since $f \in E(n_p, R)$, from (3.2.3) we get, for $k = 2, 3, \ldots,$ and $p = 1, 2, \ldots,$

$$(3.2.12) \qquad |a(n_p+k)| \leq \frac{k\, d(n_p+M) \ldots d(2)}{d(n_p+M) \ldots d(n_p+2) \cdot d(n_p+k) \ldots d(k+1)}$$

By (3.2.1), $k/(d(n_p+k) \ldots d(k+1))$ decreases with k for some $p \geq p_o$. Therefore, there is a positive integer Q_1 such that for $p \geq Q_1$ and $k \geq M+1$

$$\frac{k\, d(n_p+M) \ldots d(2)}{d(n_p+k) \ldots d(k+1)} < 1 .$$

Thus, for $p \geq \max\{p_o, Q_1\}$ and $k \geq M+1$, the inequality in (3.2.12) implies that

$$(3.2.13) \qquad |a(n_p+k)| \leq \frac{|a(n_p+1)|}{d(n_p+M) \ldots d(n_p+2)}$$

$$< \frac{|a(n_p+1)|}{d(n_p)^{M-1}}$$

Also, there is a positive integer Q_2 such that for $p > Q_2$ and $k > 2$

$$\frac{k\, d(n_p+1) \ldots d(2)}{d(n_p+k) \ldots d(k+1)} < 1$$

Using (3.2.3) again, it follows that for $p \geq Q_2$ and $k \geq 2$

$$(3.2.14) \qquad |a(n_p+k)| < |a(n_p+1)| .$$

Since $\liminf_{p\to\infty} (n_{p+1}-n_p) \geq M$, there is a positive integer Q_3 such that for $p \geq Q_3$, $(n_{p+1}-n_p) \geq M$. Let $Q = \max\{P_0, Q_1, Q_2, Q_3\}$ and suppose $p \geq Q$. Then, $(n_{p+1}-n_p+1) \geq M+1$ for $p > Q$ and so the inequality in (3.2.13) implies that, for $p \geq Q$

$$|a(n_{p+1}+1)| < \frac{|a(n_p+1)|}{d(n_p)^{M-1}}$$

An inductive argument on p shows that for $p > Q$

$$|a(n_p+1)| < \frac{|a(n_Q+1)|}{\{d(n_Q)\cdots d(n_{p-1})\}^{M-1}}$$

$$= \frac{A_2}{\{d(n_1)d(n_2)\cdots d(n_{p-1})\}^{M-1}}$$

Where 'A_2' is a constant. If $2 \leq k \leq n_{p+1}-n_p+1$, then using the above inequality (3.2.14), we get, for $p > Q$

$$(3.2.15) \qquad |a(n_p+k) < \frac{A_2}{\{d(n_1)d(n_2)\cdots d(n_{p-1})\}^{M-1}}.$$

Thus,

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k}$$

$$= \lim\sup\{|a(n_p+k)|^{1/(n_p+k)} : 2 \leq k \leq n_{p+1}-n_p+1 ; p > Q\}$$

$$\leq \frac{1}{\liminf_{p\to\infty} \{d(n_1)d(n_2)\cdots d(n_{p-1})\}^{(M-1)/n_{p+1}}}.$$

This completes the proof of Theorem 3.2.3.

Corollary 3.2.3 *Let f be in* $E(n_p, R)$ *and the sequence* $\{d_n\}_{n=1}^{\infty}$ *in* (3.1.1) *satisfies* (3.2.1). *If* $\lim_{p\to\infty}(n_{p+1}-n_p) = \infty$ *and*

$$\liminf_{p\to\infty}\{d(n_1)d(n_2)\cdots d(n_{p-1})\}^{1/n_{p+1}} > 1 \qquad (3.2.16)$$

then f is entire.

Proof. Since $\lim_{p\to\infty}(n_{p+1}-n_p) = \infty$, it follows easily from (3.2.11) that $R = \infty$, i.e., f is entire.

If $f \in E_c(n_p, R)$, the condition (3.2.1) is not needed to deduce (3.2.11).

Theorem 3.2.4 *Let f be in* $E_c(n_p, R)$. *If there is a positive integer* $M > 1$ *such that* $\liminf_{p\to\infty}(n_{p+1}-n_p) \geq M$, *then* (3.2.11) *holds.*

Proof. From (3.2.8), we get for $p = 1, 2, \ldots$ and $k = 2, 3, \ldots$,

$$|a(n_p+k)| \leq \frac{d(n_p+M)\cdots d(2)|a(n_p+1)|}{d(n_p+M)\cdots d(n_p+2)\,d(n_p+k)\cdots d(k+1)}.$$

The proof of the theorem is easily constructed by noting that $1/(d(n_p+k)\cdots d(k+1))$ is a decreasing function of k for all p and by following the same lines of proof as in Theorem 3.2.3.

Corollary 3.2.4 *Let f be in* $E_c(n_p, R)$. *If* $\lim_{p\to\infty}(n_{p+1}-n_p) = \infty$ *and* (3.2.16) *holds then f is entire.*

The proof of corollary 3.2.4 is immediate and we omit it.

Remarks. 1. We note that the conditions (3.2.1) and (3.2.16) are independent. Let $n_p = [p \log\log(p+c)]+1$, $c = 10^{10}$ and $d_n = \log(n+1)$, $n = 1, 2, \ldots$. Then, $\lim_{p\to\infty} (n_{p+1}-n_p) = \infty$ and $\liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{1/n_{p+1}} \geq e > 1$. But, (3.2.1) is not satisfied. Thus, (3.2.16) need not imply (3.2.1). To see that, (3.2.16) does not necessarily follow from (3.2.1), we may take $n_p = [e^p]$, $p = 1, 2, \ldots$ and $d_n = n$, so that (3.2.1) is clearly true and $\lim_{p\to\infty} (n_{p+1}-n_p) = \infty$. But, in this case, $\liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{1/n_{p+1}} = 1$.

2. The conditions on the sequence $\{d_n\}_{n=1}^{\infty}$ and the sequence $\{n_p\}_{p=1}^{\infty}$ in Corollary 3.2.1 and Corollary 3.2.3 are independent. To see this, take $n_p = 1+[p \log p]$, $p = 1, 2, \ldots$ and $d_n = n^a (a > 0)$, $n = 1, 2, \ldots$. Clearly, the sequence $\{d_n\}_{n=1}^{\infty}$ satisfies (3.2.1) for some $p > p_1$. Further, $(n_{p+1}-n_p) > \frac{1}{2}\log p$ for large p, so that $\lim_{p\to\infty} (n_{p+1}-n_p) = \infty$. But $(n_{p+1}-n_p) \neq o(\log d(n_p))$. This gives that the condition in corollary 3.2.1 does not follow from the condition in Corollary 3.2.3. The converse part can be seen to hold by taking any sequence satisfying (3.2.1) and (3.2.16) and by choosing the sequence $\{n_p\}_{p=1}^{\infty}$ such that $\lim_{p\to\infty} (n_{p+1}-n_p) = \infty$.

The above choices of the sequence $\{d_n\}_{n=1}^{\infty}$ and $\{n_p\}_{p=1}^{\infty}$ also show that the conditions in Corollary 3.2.2 and corollary 3.2.4 are independent.

3. Let $\lim_{n \to \infty} d_n^{1/n} = b$, $o < b < \infty$. Then, the inequality in (3.2.11) can be written as

$$(3.2.17) \quad \liminf_{p \to \infty} \{d(n_1)d(n_2)\ldots d(n_p)\}^{(M-1)/n_p} \leq b^{2(M-1)} R .$$

4. If $f \in E(n_p, R)$ and $d_n \equiv n$, then a result of Shah and Trimble [83] follows from Theorem 3.2.3. For $f \in E_c(n_p, R)$, Theorem 3.2.4 gives a new result even for $d_n = n$, $n = 1, 2, \ldots$.

3.3 In this section, we estimate the order of a function in $E(n_p, R)$ by restricting the growth of the sequence $\{n_p - n_{p-1}\}_{p=2}^{\infty}$ by slowly oscillating functions.

A function $s(x)$, continuous on $[1, \infty)$, is said to be Slowly Oscillating, (abbreviated as S.O.), if for every positive number $c > o$, that, as $n \to \infty$,

$$\lim_{x \to \infty} \frac{s(cx)}{s(x)} = 1.$$

A function $\sigma(x)$ is said to be the restriction of a slowly oscillating function $s(x)$ if $\sigma(n) = s(n)$ for every psotive integer n. It is known [35] that

$$(3.3.1) \qquad \sum_{i=1}^{n} \sigma(i) \sim n\sigma(n).$$

We now prove

<u>Theorem</u> 3.3.1 <u>Let</u> f <u>be in</u> $E(n_p, R)$. <u>Let</u> d_n's <u>in</u> (3.1.1) <u>be the restriction of a continuously differentiable function</u> $d(x)$ <u>on positive integers such that</u> $xd'(x)/d(x)$ <u>is a non-increasing function of</u> x, <u>where</u> $d'(x)$ <u>denotes the</u>

derivative of $d(x)$ with respect to x. Suppose φ and θ are S.O. functions such that $\theta(x) \to \infty$ as $x \to \infty$ and for $p = 1, 2, \ldots$

$$1 \leq \varphi(p) \leq n_p - n_{p-1} \leq \theta(p). \tag{3.3.2}$$

If

$$\alpha^* = \liminf_{p \to \infty} \frac{\log d(n_p)}{\log n_p} \tag{3.3.3}$$

$$\beta^* = \limsup_{p \to \infty} \frac{\theta(p) \log d[[\theta(p)]]}{\varphi(p) \log p} \tag{3.3.4}$$

and $\alpha^* > \beta^*$, then f is an entire function of order atmost $1/(\alpha^* - \beta^*)$. In (3.3.4), $[[\theta(p)]]$ denote the greatest integer not greater than $\theta(p)$.

We need the following Lemma.

Lemma 3.3.1 Let f be in $E(n_p, R)$ and d_n's in (3.1.1) be the restriction of a continuously differentiable function $d(x)$ on positive integer such that $xd'(x)/d(x)$ is a non-increasing function of x. Suppose θ is a real valued function defined on the set of positive integers such that $\theta(p) \to \infty$ as $p \to \infty$. If $(n_{p+1} - n_p) \leq \theta(p)$ and $1 \leq k \leq n_{p+1} - n_p$; $p \geq 2$, then for sufficiently large p,

$$\frac{\log |a(n_p+k)|}{(n_p+k)} < \log n_{p+1} \left\{ o(1) - \frac{\log d(n_{p+1})}{\log n_{p+1}} + \frac{\theta(p+1) \log d[[\theta(p+1)]]}{n_{p+1} \log n_{p+1}} + \frac{1}{n_p \log n_p} \sum_{i=2}^{p} \theta(i) \log d[[\theta(i)+1]] \right\} . \tag{3.3.5}$$

Proof. For $k = n_{p+1}-n_p+1$, (3.2.3) becomes

$$|a(n_{p+1}+1)| < \frac{(n_{p+1}-n_p+1)d(n_{p+1}-n_p+1)\dots d(2)d(n_p+1)\dots d(2)|a(n_p+1)|}{d(n_{p+1}+1)\dots d(2)}$$

An inductive argument on p in the above inequality shows that for $p \geqq 2$

$$(3.3.6)\quad |a(n_p+1)| < \frac{d(n_1+1)\dots d(2)|a(n_1+1)}{d(n_p+1)\dots d(2)} \times$$

$$\times \prod_{i=2}^{p}(n_i-n_{i-1}+1)d(n_i-n_{i-1}+1)\dots d(2).$$

Using (3.3.6) in (3.2.3), we get for $p \geqq 2$ and $1 \leqq k \leqq n_{p+1}-n_p$,

$$(3.3.7)\quad |a(n_p+k)| < \frac{A_3 k\, d(k)\dots d(2)}{d(n_p+k)\dots d(2)} \times$$

$$\times \prod_{i=2}^{p}(n_i-n_{i-1}+1)d(n_i-n_{i-1}+1)\dots d(2)$$

where $A_3 = d(n_1+1)\dots d(2)|a(n_1+1)|$, is a constant. Thus, for sufficiently large p,

$$(3.3.8)\quad \log|a(n_p+k)| < O(1)+\log k + \sum_{i=2}^{k}\log d(i) - \sum_{i=2}^{n_p+k}\log d(i) +$$

$$+ \sum_{i=2}^{p}\log(n_i-n_{i-1}+1) + \sum_{i=2}^{p}\log(d(n_i-n_{i-1}+1)\dots d(2)).$$

Since, $xd'(x)/d(x)$ is a non-increasing function of x, $xd'(x)/d(x)$ tends to λ_o, $o < \lambda_o < \infty$. as $x \to \infty$. This implies [76, pp.7] that $x^{-\lambda_o}d(x)$ is a slowly oscillating function. On applying the definition of a S.O. function, it follows that $\log d(x)$ is a S.O. function. Therefore, by (3.3.1),

$$\sum_{i=2}^{n_p+k} \log d(i) \sim (n_p+k) \log d(n_p+k) \text{ , as } p \to \infty$$

Using this asymptotic relation in (3.3.7), we get for sufficiently large p,

$$(3.3.9) \quad \frac{\log|a(n_p+k)|}{(n_p+k)} < O(1) + \frac{k \log d(k)}{(n_p+k)} - \log d(n_p+k)$$
$$+ \frac{1}{n_p+2} \sum_{i=2}^{p} \log(n_i - n_{i-1}+1)$$
$$+ \frac{1}{n_p} \sum_{i=2}^{p} (n_i - n_{i-1}) \log d(n_i - n_{i-1}+1) \text{ .}$$

Now consider the function g defined by

$$g(x) = \frac{x \log d(x)}{(n_p+x)} - \log d(n_p+x) \text{ .}$$

Since, $xd'(x)/d(x)$ is a non-increasing function of x

$$g'(x) = \frac{n_p \log d(x) + (n_p+x)\left(x \dfrac{d'(x)}{d(x)} - (n_p+x) \dfrac{d'(n_p+x)}{d(n_p+x)}\right)}{(n_p+x)^2} \geq 0 \text{ .}$$

Therefore, $g(x)$ is a non-decreasing function of x, so that for $1 \leq k \leq n_{p+1} - n_p$

$$(3.3.10) \quad \frac{k \log d(k)}{(n_p+k)} - \log d(n_p+k)$$
$$\leq \frac{(n_{p+1}-n_p) \log d(n_{p+1}-n_p)}{n_{p+1}} - \log d(n_{p+1}) \text{ .}$$

Thus the estimate of $\log|a(n_p+k)|/(n_p+k)$ with (3.3.2) and (3.3.10) becomes

$$\frac{\log|a(n_p+k)|}{(n_p+k)} < \log n_{p+1}\left\{o(1) - \frac{\log d(n_{p+1})}{\log n_{p+1}} + \frac{\theta(p+1)\log d[\![\theta(p+1)]\!]}{n_{p+1}\log n_{p+1}}\right.$$
$$\left. + \frac{1}{n_p \log n_p}\sum_{i=2}^{p} \theta(i)\log d[\![\theta(i)+1]\!]\right\}.$$

This proves the lemma .

<u>Proof of Theorem</u> 3.3.1 Using (3.3.2) , we have $n_p \geq \sum_{i=2}^{p} \varphi(i)+n_1$. Since φ is S.O. (3.3.1) gives

$$\sum_{i=2}^{p} \varphi(i) \sim p\,\varphi(p)\ ,\ \text{as } p \to \infty .$$

Thus , as $p \to \infty$,

(3.3.11) $$n_p \log n_p \geq (1+o(1))\, p\,\varphi(p) \log p.$$

The definition of α in (3.3.3) and (3.3.11) give , for sufficiently large p ,

(3.3.12) $$\frac{\theta(p+1)\log d[\![\theta(p+1)]\!]}{n_{p+1}\log n_{p+1}} = o(1) .$$

From the proof of Lemma 3.3.1 , the hypothesis $xd'(x)/d(x)$ is a non-increasing function implies that $\log d(x)$ is a S.O. function . Since $\theta(x)$ is S.O. and $\theta(x) \to \infty$ as $x \to \infty$, the function $\log d[\![\theta(p)]\!]$ is a restriction of a S.O. function on positive integers. Therefore $\sum_{i=2}^{p} \theta(i) \log d[\![\theta(i)+1]\!]$ is a restriction of a S.O. function on positive integers.

Consequently , by (3.3.1) as $p \to \infty$

$$\sum_{i=2}^{p} \theta(i) \log d[[\theta(i)+1]] \sim p\,\theta(p) \log d[[\theta(p)+1]]$$
$$\sim p\,\theta(p) \log d[[\theta(p)]]$$

Using this asymptotic relation , (3.3.11) and (3.3.12) ,

we obtain for large p

$$\frac{1}{n_p \log n_p} \sum_{i=2}^{p} \theta(i) \log d[[\theta(i)+1]] + \frac{\theta(p+1) \log d[[\theta(p+1)]]}{n_{p+1} \log n_{p+1}}$$

$$\leq \frac{(1+o(1))p\,\theta(p) \log d[[\theta(p)]]}{p\,\varphi(p) \log p} + o(1)$$

$$= \frac{\theta(p) \log d[[\theta(p)]]}{\varphi(p) \log p} + o(1).$$

Lemma 3.3.1 together with the above estimate shows that for $p \geq 2$ and $2 \leq k \leq n_{p+1} - n_p + 1$

$$(3.3.13) \quad \frac{\log |a(n_p+k)|}{(n_p+k)} < \log n_{p+1} \left\{ o(1) - \frac{\log d(n_p)}{\log n_p} + \frac{\theta(p) \log d[[\theta(p)]]}{\varphi(p) \log p} \right\}$$

Since $\alpha^* > \beta^*$, we get from the above inequality that

$\limsup_{k \to \infty} \frac{\log |a_k|}{k} = -\infty$ and therefore , f is an entire function . Further , if Λ is the order of f , then the coefficient characterization of Λ (c.f. section 1.3) and (3.3.13) easily give $\Lambda \leq 1/(\alpha^* - \beta^*)$. This completes the proof of the theorem.

<u>Remark</u>. Putting $d_n = n$, n = 1, 2, .. in Theorem 3.3.1 , we get a result obtained earlier by Shah and Trimble [83].

3.4 This section is devoted to finding estimates for the type of entire functions of finite order belonging to th classes $E(n_p; \infty)$ and $E_c(n_p; \infty)$.

Theorem 3.4.1 Let $f \in E(n_p; \infty)$ have order Λ, $0 < \Lambda < \infty$ and Let d_n's in (3.1.1) satisfies (3.2.1). Then,

$$(3.4.1) \quad \liminf_{p \to \infty} \left[\frac{\{d(n_1)d(n_2)\ldots d(n_p)\}^{1/n_{p+1}}}{n_{p+1}^{1/\Lambda}} \right]$$

$$\leq \frac{\limsup_{p \to \infty} (2d_2)^{\Lambda p/n_p}}{e\Lambda\tau} \leq \frac{(2d_2)^{\Lambda}}{e\Lambda\tau}.$$

Proof. Since $f(z) = \sum_{n=0}^{\infty} a_n z^n \in E(n_p; \infty)$ and d_n's satisfies (3 the inequality in (3.2.5) is satisfied. Now, the coefficien characterization (1.3.9) of τ gives

$$e\Lambda\tau = \limsup_{k \to \infty} k|a_k|^{\Lambda/k}$$

$$= \limsup \{(n_p+k)|a(n_p+k)|^{\Lambda/(n_p+k)};\ 2 \leq k \leq n_{p+1}-n_p+1;\ p >$$

$$\leq \frac{\limsup_{p\to\infty} (2d_2)^{\Lambda p/n_p}}{\liminf_{p \to \infty} \left[\frac{\{d(n_1)d(n_2)\ldots d(n_p)\}^{1/n_{p+1}}}{n_{p+1}^{1/\Lambda}} \right]^{\Lambda}}$$

by using the inequality (3.2.5). This proves the first inequality in (3.4.1). The second inequality in (3.4.1) is obvious.

Corollary 3.4.1 Assume the hypothesis of Theorem 3.4.1. If

$$(3.4.2) \qquad \liminf_{p\to\infty} \left[\frac{\Lambda}{n_p} \sum_{i=1}^{n_{p-1}} \log d(i) - \log n_p \right] = \infty$$

then $\tau = 0$.

Putting $d_n \equiv n$ in Theorem 3.4.1 we have

Corollary 3.4.2 Let f be an entire function of order Λ, $0 < \Lambda < \infty$, and type τ. Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $f^{(n_p)}$ is univalent in Δ. Then,

$$(3.4.3) \qquad \liminf_{p\to\infty} \left[\frac{(n_1 n_2 \cdots n_p)^{1/n_p}}{n_p^{1/\Lambda}} \right]^{\Lambda} \leq \frac{\limsup_{p\to\infty} 4^{\Lambda p/n_p}}{e\Lambda\tau} \leq \frac{4^{\Lambda}}{e\Lambda\tau}$$

If $f \in E_c(n_p, \infty)$, our next theorem gives a bound for the type τ which is better than that given by Theorem 3.4.1.

Theorem 3.4.2 Let $f \in E_c(n_p, \infty)$ have order Λ, $0 < \Lambda < \infty$ and type τ. Then,

$$(3.4.4) \qquad \liminf_{p\to\infty} \left[\frac{\{d(n_1)d(n_2)\cdots d(n_p)\}^{1/n_{p+1}}}{n_{p+1}^{1/\Lambda}} \right]^{\Lambda}$$

$$\leq \frac{\limsup_{p\to\infty} d_2^{\Lambda p/n_p}}{e\Lambda\tau} \leq \frac{d_2^{\Lambda}}{e\Lambda\tau}.$$

Proof. Since $f(z) = \sum_{n=0}^{\infty} a_n z^n \in E_c(n_p, \infty)$, we get from (3.2.15)

$$|a(n_p+k)| \leq \frac{A_2 d_2^p}{d(n_1)d(n_2)\cdots d(n_p)}$$

where A_2 is a constant . Thus , by (1.3.9) , we deduce that

$$e\Lambda\tau = \limsup_{k\to\infty} k|a_k|^{\Lambda/k}$$

$$= \lim \sup\{(n_p+k)|a(n_p+k)|^{/(n_p+k)} : 2 \leqq k \leqq n_{p+1}-n_p ; p \geqq 2\}$$

$$\leqq \frac{\limsup_{p\to\infty} d_2^{\Lambda p/n_p}}{\liminf_{p\to\infty}\left[\frac{\{(d(n_1)d(n_2)\ldots d(n_p)\}^{1/n_{p+1}}}{n_{p+1}^{1/\Lambda}}\right]^{\Lambda}}.$$

This proves the inequality in (3.4.4).

<u>Corollary</u> 3.4.3 <u>Assume the hypothesis of Theorem</u> 3.4.2. <u>If</u> (3.4.2) <u>holds then</u> $\tau = 0$.

<u>Remark</u>. If d_n's in (3.1.1) satisfy (3.2.1) , then Corollary 3.4.3 follows from Corollary 3.4.1.

Taking $d_n = n$, $n = 1, 2, \ldots$, in Theorem 3.4.2 , we have

<u>Corollary</u> 3.4.4 <u>Let</u> f <u>be an entire function of order</u> Λ, $0 < \Lambda < \infty$ <u>and type</u> τ. <u>Let</u> $\{n_p\}_{p=1}^{\infty}$ <u>be a strictly increasing sequence of positive integers such that</u> $f^{(n_p)}$ <u>is convex univalent in</u> Δ . <u>Then</u> ,

$$(3.4.5)\quad \liminf_{p\to\infty}\left[\frac{(n_1 \cdot n_2 \cdot \ldots n_p)^{1/n_p}}{n_p^{1/\Lambda}}\right]^{\Lambda} \leqq \frac{\limsup_{p\to\infty} 2^{\Lambda p/n_p}}{e\Lambda\tau} \leqq \frac{2^{\Lambda}}{e\Lambda\tau}.$$

The next theorem shows that if $d_n = n$, $n = 1, 2, \ldots$, and $\limsup_{p\to\infty} (n_p - n_{p-1}) < \infty$ then $\tau = 0$.

<u>Theorem 3.4.3</u> <u>Let f be an entire function of order $\Lambda > 1$ and type τ. Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $f^{(n_p)}$ is univalent in Δ. If $\limsup_{p\to\infty} (n_p - n_{p-1}) < \infty$, then</u> $\tau = 0$.

<u>Proof</u>. Since $f^{(n_p)}$ is univalent in Δ, we have from (3.2.3) for $k = 2, 3, \ldots$ and $p = 2, 3, \ldots$

$$|a(n_p+k)| \le \frac{k\, k!\,(n_p+1)!\,|a(n_p+1)|}{(n_p+k)!} \tag{3.4.6}$$

where, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Letting $k = n_{p+1} - n_p + 1$ and inducting on p, we get for $p = 2, 3, \ldots$

$$|a(n_p+1)| \le \frac{A_1}{(n_p+1)!} \prod_{i=2}^{p} (n_i - n_{i-1}+1)(n_i - n_{i-1}+1)!$$

where, $A_1 = (n_1+1)!\,|a(n_1+1)|$ is a constant. Combining the above inequality in (3.4.6), we conclude that for $2 \le k \le n_{p+1} - n_p + 1$ and $p \ge 2$.

$$|a(n_p+k)| \le \frac{A_1 k\, k!}{(n_p+k)!} \prod_{i=2}^{p} (n_i - n_{i-1}+1)(n_i - n_{i-1}+1)!\,. \tag{3.4.7}$$

If n is a positive integer greater than 1, then we can find two positive numbers, k_1 and k_2, such that

$$k_1 n^{\frac{1}{2}} \left(\frac{n}{e}\right)^n < n! < k_2 n^{\frac{1}{2}} \left(\frac{n}{e}\right)^n$$

Further, if $\{n_p\}_{p=1}^{\infty}$ is a strictly increasing sequence of non-negative integers, then for each $p \ge 2$, it is known that [85]

$$\prod_{i=2}^{p} (n_i - n_{i-1}+1)^{1/(n_p+2)} \le \left(1 + \frac{n_p}{p}\right)^{p/n_p} \le 2\,. \tag{3.4.8}$$

Using this on the right side of (3.4.7), taking the (n_p+k)th root of both sides of the resulting inequality, and applying (3.4.8), to part of right side of this, it follows that for $2 \leqq k \leqq n_{p+1}-n_p+1$ and $p \geqq 2$,

$$(3.4.9) \qquad |a(n_p+k)|^{1/(n_p+k)} \leqq 2\left[\frac{Ak_2^p e^{n_1+1-p}}{k_1}\left(\frac{k^3}{n_p+k}\right)^{\frac{1}{2}}\right]^{1/(n_p+k)}$$
$$\left[\frac{k^{k/(n_p+k)}}{n_p+k} \prod_{i=2}^{p} (n_i-n_{i-1}+1)^{(n_i-n_{i-1})/(n_p+k)}\right]$$

Now we maximize the expression in the second set of brackets on the right side of (3.4.9). Let

$$a = \prod_{i=2}^{p} (n_i-n_{i-1}+1)^{n_i-n_{i-1}}$$

and $b = n_p$. Define $\tilde{\varphi}$ for $x > o$ by

$$\tilde{\varphi}(x) = \frac{(ax^x)^{1/(b+x)}}{b+x}$$

Since the minimum value of $\tilde{\varphi}$ is attained at $x = a^{1/b}$, it follows that the maximum value of $\tilde{\varphi}$ on any closed interval occurs at one of the end points. Hence, for $p \geqq 2$ and $2 \leqq k \leqq n_{p+1}-n_p+1$,

$$\frac{k^{k/n_p+k}}{(n_p+k)} \prod_{i=2}^{p} (n_i-n_{i-1}+1)^{(n_i-n_{i-1})/(n_p+k)}$$

$$\leqq \max\left\{\frac{4^{1/(n_p+2)}}{n_p+2} \prod_{i=2}^{p}(n_i-n_{i-1}+1)^{(n_i-n_{i-1})/(n_p+2)},\right.$$
$$\left.\frac{(n_{p+1}-n_p+1)^{1(n_{p+1}+1)}}{n_{p+1}+1} \prod_{i=2}^{p}(n_i-n_{i-1}+1)^{(n_i-n_{i-1})/(n_{p+1}+1)}\right\}$$

After simplifying this, and then using it on the right side of (3.4.9), we have that for $p \geq 2$ and $2 \leq k \leq n_{p+1} - n_p + 1$,

$$(3.4.10) \quad |a(n_p+k)|^{1/(n_p+k)} \leq 2\left[\frac{A_1 k_2^p e^{n_1+1-p}}{k_1}\left(\frac{k^3}{n_p+k}\right)^{\frac{1}{2}}\right]^{1/(n_p+k)} \times$$

$$\times \quad \max\left\{ \frac{4}{n_p}^{1/n_p} \prod_{i=2}^{p} (n_i - n_{i-1}+1)^{(n_i-n_{i-1})/n_p} , \right.$$

$$\left. , \frac{n_{p+1}^{1/n_{p+1}}}{n_{p+1}} \prod_{i=2}^{p+1} (n_i - n_{i-1}+1)^{(n_i-n_{i-1})/n_{p+1}} \right\}$$

Since

$$e \Lambda \tau = \limsup_{k \to \infty} k |a_k|^{\Lambda/k}$$

$$= \lim \sup \{ (n_p+k) | a(n_p+k)|^{\Lambda/(n_p+k)} : 2 \leq k \leq n_{p+1} - n_p + 1, p \geq 2 \}$$

$$(3.4.11) \qquad \leq \limsup_{p \to \infty} \frac{(n_{p+1}+1)}{n_p^{\Lambda}} \prod_{i=2}^{p} (n_i - n_{i-1}+1)^{\Lambda(n_i-n_{i-1})/n_p}$$

by using the inequality in (3.4.10). Here A_1 is a constant. The hypothesis $\limsup_{p\to\infty} (n_p - n_{p-1}) < \infty$ gives $n_p \sim n_{p+1}$ as $p \to \infty$. Thus (3.4.11) becomes,

$$(3.4.12) \quad e \Lambda \tau \leq A_1 \limsup_{p \to \infty} \frac{1}{n_p^{\Lambda-1}} \prod_{i=2}^{p} (n_i - n_{i-1}+1)^{\Lambda(n_i-n_{i-1})/n_p}$$

If $\{x_p\}_{p=1}^{\infty}$ is a strictly increasing sequence of positive numbers such that $\lim_{p\to\infty} x_p = \infty$, and if $\{y_p\}_{p=1}^{\infty}$ is a sequence of real numbers, then [36]

$$\limsup_{p \to \infty} \frac{y_p}{x_p} \leq \limsup_{p \to \infty} \frac{y_p - y_{p-1}}{x_p - x_{p-1}} .$$

For applying the above inequality, we take $x_p = n_p$ and let

$$y_p = \sum_{i=2}^{p} (n_i - n_{i-1}) \log(n_i - n_{i-1} + 1) - (\Lambda - 1) n_p \log n_p \,.$$

Then,

$$\frac{y_p - y_{p-1}}{x_p - x_{p-1}} = \log(n_p - n_{p-1} + 1) + (\Lambda - 1)\left[\log\left(1 - \frac{n_{p-1} - 1}{n_p}\right) + \frac{n_{p-1}/n_p}{1 - n_{p-1}/n_p} \log \frac{n_{p-1}}{n_p}\right].$$

The second expression inside the bracket of this equation is a decreasing function of n_{p-1}/n_p. Further, $\lim\limits_{x \to 1^-} (x \log x)/(1-x) = -1$. Hence, $\limsup\limits_{p \to \infty} y_p/x_p \leq -\infty$.

Now, since

$$\frac{y_p}{x_p} = \log \frac{1}{n_p^{\Lambda - 1}} \prod_{i=2}^{p} (n_i - n_{i-1} + 1)^{\Lambda (n_i - n_{i-1})/n_p}$$

it follows from (3.4.12) that $\tau = 0$. This establishes the theorem.

The following theorems are in the other direction.

<u>Theorem</u> 3.4.4 <u>Let</u> f <u>be in</u> $E(n_p; \infty)$ <u>have order</u> Λ, $0 < \Lambda < \infty$, <u>and type</u> τ. <u>Let</u> d_n's in (3.1.1) <u>satisfies</u> (3.2.1). <u>If there is a positive integer</u> $M > 1$ <u>such that</u> $\liminf\limits_{p \to \infty} (n_p - n_{p-1}) \geq M$, <u>then</u>

$$(3.4.13) \quad \liminf_{p \to \infty} \left[\frac{(d(n_1) d(n_2) \ldots d(n_{p-1}))^{(M-1)/n_{p+1}}}{n_{p+1}^{1/\Lambda}} \right]^{\Lambda} \leq \frac{1}{e \Lambda \tau}.$$

<u>Proof</u>. From (3.2.15), we get for $p > Q$ and $2 \leq k \leq n_{p+1} - n_p + 1$,

$$|a(n_p + k)| \leq \frac{A_2}{(d(n_1) d(n_2) \ldots d(n_{p-1}))^{M-1}}.$$

Now, by using (1.3.9), we obtain

$$e\Lambda T = \lim\sup\{(n_p+k)|a(n_p+k)|^{\Lambda/(n_p+k)} : 2 \le k \le n_{p+1}-n_p+1;\ p > Q\}$$

$$\le \frac{1}{\liminf\limits_{p\to\infty}\left[\dfrac{(d(n_1)d(n_2)\cdots d(n_{p-1}))^{(M-1)/n_{p+1}}}{n_{p+1}^{1/\Lambda}}\right]^{\Lambda}}$$

This proves (3.4.13).

<u>Corollary</u> 3.4.6 <u>Assume the hypothesis of Theorem</u> 3.4.4. If $\lim\limits_{p\to\infty}(n_p-n_{p-1}) = \infty$ <u>and</u>

$$(3.4.14)\qquad \liminf_{p\to\infty}\left[\frac{(d(n_1)d(n_2)\cdots d(n_{p-1}))^{1/n_{p+1}}}{n_{p+1}^{1/\Lambda}}\right]^{\Lambda} > 1$$

then $\tau = 0$.

For $d_n \equiv n$, $n = 1, 2, \ldots$, Theorem 3.4.4 gives

<u>Corollary</u> 3.4.7 <u>Let</u> f <u>be an entire function of order</u> Λ, $0<\Lambda<\infty$, <u>and type</u> τ. <u>Let</u> $\{n_p\}_{p=1}^{\infty}$ <u>be a strictly increasing sequence of positive integers such that</u> $f^{(n_p)}$ <u>is univalent in</u> Δ. <u>If there is a positive integer</u> $M > 1$ <u>such that</u> $\liminf\limits_{p\to\infty}(n_p-n_{p-1}) \ge M$, <u>then</u>

$$(3.4.15)\qquad \liminf_{p\to\infty}\left[\frac{(n_1 n_2\cdots n_{p-1})^{(M-1)/n_p}}{n_p^{1/\Lambda}}\right]^{\Lambda} \le \frac{1}{e\Lambda\tau}.$$

If $f \in E_c(n_p;\infty)$; the condition (3.2.1) is not needed to deduce the estimate in (3.4.13)

Theorem 3.4.5 Let f be in $E_c(n_p; \infty)$ have order Λ, $0 < \Lambda < \infty$ and type τ. If there is a positive integer $M > 1$ such that $\liminf_{p \to \infty} (n_p - n_{p-1}) \geq M$, then (3.4.13) holds.

Proof. The proof is same as that of Theorem 3.4.4.

Corollary 3.4.8 Assume the hypothesis of Theorem 3.4.5. If $\lim_{p \to \infty} (n_p - n_{p-1}) = \infty$ and (3.4.14) holds, then $\tau = 0$.

CHAPTER IV

GROWTH OF ENTIRE FUNCTIONS WITH SOME UNIVALENT GELFOND-LEONTEV DERIVATIVES

4.1 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Let $\Psi(z) = \sum_{n=0}^{\infty} e_n z^n$, $e_0 = 1$, $e_n = (d_n \cdots d_1)^{-1}$ where $\{d_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive numbers. The function $\Psi(z)$ is called a comparision function (c.f. Section 1.3) if $e_{n+1}/e_n \to 0$. Thus, a comparision function is necessarily entire. When $\Psi(z)$ is a comparision function, Nachbin defined Ψ-type of an entire function (c.f. (1.3.16)). The concept of Ψ-type is an extension of the notion of exponential type of f.

We define the Ψ-order of an entire function f as the infimum of all non-negative numbers A such that for sufficiently large values of r and some constant M_1,

$$|f(re^{i\theta})| \leq M_1 \ \Psi(Ar), \quad z = re^{i\theta}. \tag{4.1.1}$$

We shall denote the Ψ-order of f by $\rho_\Psi(f)$. If $\Psi(z) = e^z$, then the Ψ-order $\rho_\Psi(f)$ of the entire function f concides with its classical order (c.f. section 1.3).

The characterization of Ψ-type $\tau_\Psi(f)$ of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in terms of the coefficient a_n is given by (1.3.16). Thus,

$$\tau_\Psi(f) = \limsup_{n \to \infty} \left|\frac{a_n}{e_n}\right|^{\frac{1}{n}}.$$

The above coefficient characterization motivates the definition of Ψ-type of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\{z: |z| < R\}$. Thus (c.f. (1.5.4)), the Ψ-type of $f \in H_R$ is defined as

$$\tau_{\Psi}(f) = \limsup_{n \to \infty} \left|\frac{a_n}{e_n}\right|^{\frac{1}{n}}$$

However, an analogous coefficient characterization of Ψ-order of an entire function is not known.

In Section 4.2, we consider functions in the class H_R having finite Ψ-type and first find a relation between the radii of convexity $\rho_n(c)$ of the Gelfond-Leontev derivatives $D^{n_p}f$ and the radius of convergence R of f. Further, in this section, we obtain the growth of radii of univalence ρ_n and radii of convexity $\rho_n(c)$ (c.f. Section 1.5) of $D^n f$ in terms of the growth numbers γ and δ (c.f. Section 1.3) of an entire function f. In Section 4.3 we find an estimate for the Ψ-type of an entire function in terms of the exponents n_p for which $D^{n_p}f$ is analytic and univalent in Δ. Section 4.4 is devoted to the study of Ψ-order of an entire function in relation to univalent functions with some analytic and univalent Gelfond-Leontev derivatives. In this Section, we first find a characterization for Ψ-order of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in terms of the coefficients a_n. Using this characterization an upper bound for Ψ-order of an entire function is found when some of the Gelfond-Leontev derivatives are assumed to be analytic and univalent in Δ. By taking $d \equiv n$, some of the results of

Shah [80], Shah and Trimble [86, 88] follow from our results.

4.2 In this section, for f in the class H_R, $o < R \leq \infty$, we find a relation between the radii of convexity $\rho_n(c)$ of Gelfond-Leontev derivatives $D^n f$ and the radius of convergence R of f. If f is an entire function, the relations between the radii of univalence ρ_n and radii of convexity $\rho_n(c)$ of $D^n f$ with the growth numbers γ and δ of f are also obtained.

We prove

<u>Theorem</u> 4.2.1 <u>Let</u> $f(z) = \sum_{n=o}^{\infty} a_n z^n$ <u>have radius of convergence</u> R, $o < R \leq \infty$, <u>is of finite</u> Ψ<u>-type and</u> $\rho_n(c)$, $n = 1, 2, \dots$ <u>be the radius of convexity of the Gelfond-Leontev derivative</u> $D^n f$ <u>of</u> f. <u>Suppose</u> $\lim_{n\to\infty} |a_n/a_{n+1}|$ <u>exists</u>. <u>Then</u>,

$$\limsup_{n\to\infty} d_n \rho_n(c) \leq \sqrt{\left(\frac{d_3}{4d_3-3d_2}\right)}\, d_2 R \tag{4.2.1}$$

<u>Proof</u>. If $R = \infty$, then (4.2.1) is trivially true. So, let $o < R < \infty$. Since $f \in H_R$ is of finite Ψ-type, we get from (1.5.4) that $r_o(\Psi) = \sup_{n\geq 1}\{d_n\} < \infty$ and so $d_n \sim d_{n+1}$, as $n \to \infty$. The function K_n, defined by

$$K_n(z) = \frac{D^n f(\rho_n(c) z) - D^n f(o)}{\rho_n(c) D^{n+1} f(o)}$$

$$= z + \frac{d_{n+2}}{d_2}\,\frac{a_{n+2}}{a_{n+1}}\rho_n(c) z^2 + \frac{d_{n+2}d_{n+3}}{d_2 d_3}\,\frac{a_{n+3}}{a_{n+1}}\,\rho_n^2(c) z^3 + \dots$$

is analytic and convex in Δ. Therefore, by using (1.2.11), we get

$$\left|\left(\frac{d_{n+2}}{d_2}\frac{a_{n+2}}{a_{n+1}}\rho_n(c)\right)^2 - \frac{d_{n+2}d_{n+3}}{d_2 d_3}\frac{a_{n+3}}{a_{n+1}}\rho_n^2(c)\right| \leq \frac{1}{3}\left(1-\left|\frac{d_{n+2}}{d_2}\frac{a_{n+2}}{a_{n+1}}\right|^2\rho_n^2(c)\right)$$

or,

$$4\left(\frac{d_{n+2}}{d_2}\right)^2\left|\frac{a_{n+2}}{a_{n+1}}\right|^2\rho_n^2(c) - 3\,\frac{d_{n+2}d_{n+3}}{d_2 d_3}\left|\frac{a_{n+3}}{a_{n+1}}\right|\rho_n^2(c) \leq 1$$

Since $d_n \sim d_{n+1}$ as $n \to \infty$ and $\lim_{n\to\infty} |a_n/a_{n+1}|$ exists, taking limit superior in the above inequality, we deduce that

$$\limsup_{n\to\infty} d_n \rho_n(c) \leq \left(\sqrt{\frac{d_3}{4d_3 - 3d_2}}\right) d_2 R$$

<u>Remark</u>. We observe from the proof that Theorem 4.2.1 continues to hold if the hypothesis of finiteness of Ψ-type of f is replaced by the hypothesis $d_n \sim d_{n+1}$ as $n \to \infty$ where $\{d_n\}_{n=1}^{\infty}$ is as in (1.3.14).

<u>Corollary</u> 4.2.1 <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^n$ <u>have radius of convergence</u> R, $0 < R \leq \infty$, <u>and is of finite</u> Ψ-<u>type</u>. <u>Suppose</u> $\lim_{n\to\infty} |a_n/a_{n+1}|$ <u>exists</u>. <u>If</u> $\lim_{n\to\infty} d_n\rho_n(c) = \infty$, <u>then</u> f <u>is a transcendental entire function</u>.

The following theorem gives relations of ρ_n and $\rho_n(c)$ with the growth numbers γ and δ of an entire function.

<u>Theorem</u> 4.2.2 <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^n$ <u>be a transcendental entire function such that</u> $|a_n/a_{n+1}|$ <u>is eventually a positive and non-decreasing sequence</u>. <u>Let</u> ρ_n <u>and</u> $\rho_n(c)$ <u>be the radii of univalence and radii of convexity of</u> $D^n f$ <u>and the sequence</u> $\{d_n\}_1^{\infty}$ <u>in</u> $D^n f$ <u>satisfies</u> $d_n \sim d_{n+1}$ as $n \to \infty$, <u>then</u>

$$(4.2.2)\quad \lim_{n\to\infty} \begin{matrix}\sup\\ \\ \inf\end{matrix} \left(\frac{d_n\rho_n}{n}\right) \le \begin{cases} d_2\sqrt{\dfrac{d_3}{d_2-d_2}}/\delta \\ \\ d_2\sqrt{\dfrac{d_3}{d_3-d_2}}/\gamma \end{cases}$$

and

$$(4.2.3)\quad \lim_{n\to\infty} \begin{matrix}\sup\\ \\ \inf\end{matrix} \left(\frac{\delta_n\rho_n(c)}{n}\right) \le \begin{cases} d_2\sqrt{\dfrac{d_3}{4d_3-3d_2}}/\delta \\ \\ d_2\sqrt{\dfrac{d_3}{4d_3-3d_2}}/\gamma \end{cases}$$

<u>where</u> γ <u>and</u> δ <u>are defined by</u> (1.3.12).

<u>Proof</u>. The function H_n, defined by

$$(4.2.4)\qquad H_n(z) = \frac{D^n f(\rho_n z) - D^n f(o)}{\rho_n\ D^{n+1} f(o)}$$

$$= z + \frac{d_{n+2}}{d_2}\frac{a_{n+2}}{a_{n+1}}\rho_n z^2 + \frac{d_{n+2}d_{n+3}}{d_2 d_3}\frac{a_{n+3}}{a_{n+1}}\rho_n^2 z^3 + \ldots$$

is in S. Therefore, by (1.2.1)

$$\left|\left(\frac{d_{n+2}}{d_2}\right)^2\left(\frac{a_{n+2}}{a_{n+1}}\rho_n\right)^2 - \frac{d_{n+2}\ d_{n+3}}{d_2 d_3}\frac{a_{n+3}}{a_{n+1}}\rho_n^2\right| \le 1 .$$

Or,

$$\left(\frac{d_n\rho_n}{n}\right)^2 \left|\ \frac{1}{d_2^2}\left(\frac{d_{n+2}}{d_n}\right)^2\left(\frac{n}{\left|\frac{a_{n+1}}{a_{n+2}}\right|}\right)^2 - \frac{1}{d_2 d_3}\frac{d_{n+2}\ d_{n+3}}{d_n^2}\frac{n^2}{\left|\frac{a_{n+2}}{a_{n+3}}\frac{a_{n+1}}{a_{n+2}}\right|}\right| \le 1 .$$

Now, by using (1.3.13) and the fact that $d_n \sim d_{n+1}$ as $n \to \infty$, we deduce the inequalities in (4.2.2) by the above inequality.

The inequalities in (4.2.3) follow on replacing ρ_n by $\rho_n(c)$ in (4.2.4), applying (1.2.11), followed by (1.3.13) and proceeding as above.

Remark 1. If the condition $d_n \sim d_{n+1}$ as $n \to \infty$ is dropped from Theorem 4.2.2, then the following inequalities hold instead of (4.2.2) and (4.2.3):

$$(4.2.5) \qquad \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(\frac{d_n\rho_n}{n}\right) \leq \begin{cases} \dfrac{2d_2}{\delta} \\[2ex] \dfrac{2d_2}{\gamma} \end{cases}$$

and

$$(4.2.6) \qquad \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(\frac{d_n\rho_n(c)}{n}\right) \leq \begin{cases} \dfrac{d_2}{\delta} \\[2ex] \dfrac{d_2}{\gamma} \end{cases}$$

The proof of (4.2.5) and (4.2.6) is easily constructed as follows.

Since the function H_n, defined by (4.2.4), is in S, we have by (1.1.2)

$$\left|\frac{a_{n+2}}{a_{n+1}}\right| \leq \frac{2d_2}{d_{n+2}\rho_n} .$$

This gives the inequalities in (4.2.5) on using (1.3.13) and proceeding to limit. Likewise, the inequalities in (4.2.6) follow on replacing ρ_n by $\rho_n(c)$ in (4.2.4), using (1.2.9) followed by (1.3.13).

2. We observe that, in general, the estimate found in (4.2.2) is better than the estimate in (4.2.5). However, if $3d_3 < 4d_2$, then (4.2.5) gives a better estimate than the estimate in (4.2.2). The estimate in (4.2.3) is better than the estimate in (4.2.6).

3. With $d_n \equiv n$, Theorems 4.2.1 and 4.2.2 give some of the results of Shah and Trimble [88].

4.3 The main purpose of this section is to find an upper bound for Ψ-type of an entire function assuming some of the Gelfond-Leontev derivatives to be analytic and univalent in the unit disc Δ. Throughout in the sequel, $E(n_p, R)$, $o < R \leq \infty$ denotes the class of functions, introduced in Section 3.1.

Let f be in $E(n_p, 1)$. Set, for $p = 1, 2, \ldots$,

$$(4.3.1)\qquad \left\{\begin{aligned} &\sum_{i=2}^{p} \log(d(n_i - n_{i-1}+1)\cdots d(2)) = \xi_p \\ &\frac{1}{n_p}\Big(\sum_{i=2}^{n_p} \log d(i) - \xi_{p+1}\Big) = \xi_p^* \end{aligned}\right.$$

and define

$$(4.3.2)\qquad \left\{\begin{aligned} &\limsup_{p\to\infty} \frac{\xi_p}{n_p} = \eta \\ &\lim_{p\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{p}{n_p} = \begin{matrix}\tilde{\theta}_1\\ \tilde{\theta}_2\end{matrix} . \end{aligned}\right.$$

In the following theorem we find a condition involving ξ_p^* which is weaker than the condition $(n_{p+1}-n_p) = o(\log d(n_p))$ as $p \to \infty$ (c.f. Corollary 3.2.1) but still forces the function f to be entire. This is needed in the sequel to construct an example concerning Theorem 4.3.2 wherein the desired upper bound on Ψ-type is found.

<u>Theorem</u> 4.3.1 <u>Let</u> $f \in E(n_p, 1)$. <u>If</u> $\lim\limits_{p\to\infty} \xi_p^* = \infty$, <u>then</u> f <u>is entire</u>.

<u>Proof</u>. Since, for $2 \leq k \leq n_{p+1}-n_p+1$,

$$\left(\frac{d_k \cdots d_2}{d_{n_p+k} \cdots d_2}\right)^{1/(n_p+k)} \leq \left(\frac{d(n_{p+1}-n_p+1) \cdots d(2)}{d(n_p+2) \cdots d(2)}\right)^{1/(n_p+2)}$$

we deduce from (3.3.7) that for sufficiently large p

$$|a(n_p+k)|^{1/(n_p+k)} \leq \frac{1+o(1)}{(d(n_p+2)\cdots d(2))^{1/(n_p+2)}} \left(\prod_{i=2}^{p} (n_i-n_{i-1}+1)\right)^{1/(n_p+2)} \times$$

$$\times \left(\prod_{i=2}^{p+1} d(n_i-n_{i-1}+1)\cdots d(2)\right)^{1/(n_p+2)}$$

Now, by using (3.4.8) we obtain for sufficiently large p

$$|a(n_p+k)|^{1/(n_p+k)} \leq \frac{2(1+o(1))}{(d(n_p)\cdots d(2))^{1/n_p}} \left(\prod_{i=2}^{p+1} (d(n_i-n_{i-1}+1)\cdots d(2))\right)^{1/n_p}$$

Hence ,

$$(4\cdot 3\cdot 3) \quad \frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k}$$

$$= \limsup \{ |a(n_p+k|^{1/(n_p+k)} ; 2 \leq k \leq n_{p+1}-n_p+1 ; p > 0 \}$$

$$\leq 2 \limsup_{p\to\infty} \frac{\prod_{i=2}^{p+1} (d(n_i-n_{i-1}+1)\cdots d(2))^{1/n_p}}{(d(n_p)\cdots d(2))^{1/n_p}}$$

Now,

$$\frac{\prod_{i=2}^{p+1} (d(n_i-n_{i-1}+1)\cdots d(2))^{1/n_p}}{(d(n_p)\cdots d(2))^{1/n_p}}$$

$$= \exp\left[\frac{1}{n_p}\left(\sum_{i=2}^{p+1} \log(d(n_i-n_{i-1}+1)\cdots d(2)) - \sum_{i=2}^{n_p} \log d(i)\right)\right]$$

$$= \exp\left[-\frac{1}{n_p}\left(\xi_{p+1} - \sum_{i=2}^{n_p} \log d(i)\right)\right]$$

$$= \exp(-\xi_p^*) .$$

Since $\lim_{p\to\infty} \xi_p^* = \infty$, it follow from (4.3.3) that $R = \infty$; i.e. f is an entire function .

We show in the following example that there exists a sequence $\{n_p\}_{p=1}^{\infty}$ such that $(n_{p+1}-n_p) \neq o(\log d(n_p))$ yet , $\lim_{p\to\infty} \xi_p^* = \infty$. Thus , Theorem 4.3.1 . applies while Corollary 3.2.1 is not applicable .

<u>Example</u> 4.3.1 Let $\{M_k\}_{k=1}^{\infty}$ be any rapidly increasing sequence such that $M_{k+1} \geq k\, 2^{M_k}$, $M_1 = 10^2$. Let,

$$n(k) = k \quad , 1 \leq k \leq M_1$$
$$n(k) = 2^{M_k}+j \ , 1 \leq j \leq M_{k+1}-M_k \ ; \ k = 1, 2, \ldots$$

and

$$d(n) = n^a, \ a > 0$$

then the sequence $\{d_n\}_{n=1}^{\infty}$ satisfies (3.2.1) and

$$\lim_{p\to\infty} \left\{ \begin{matrix}\sup\\ \inf\end{matrix} \right\} \frac{n_{p+1}-n_p}{\log d(n_p)} = \begin{matrix}\infty\\ 0\end{matrix} \ ; \ \lim_{p\to\infty} \xi_p^* = \infty .$$

Next, we find an estimate for the Ψ-type of functions in the class $E(n_p, \infty)$.

<u>Theorem</u> 4.3.2 <u>Let</u> $f \in E(n_p, \infty)$. <u>Then, the</u> Ψ-<u>type of</u> f <u>satisfies</u>

$$(4.3.4) \qquad \tau_\Psi(f) \leq \exp\left[\eta\left(2-\frac{\tilde{\theta}_1}{\tilde{\theta}_2}\right) + \tilde{\theta}_1 \log\left(1+\frac{1}{\tilde{\theta}_1}\right)\right]$$

<u>where</u> η, $\tilde{\theta}_1$ <u>and</u> $\tilde{\theta}_2$ <u>are defined by</u> (4.3.2). <u>The function</u> $\tilde{\theta}_1 \log(1+\frac{1}{\tilde{\theta}_1})$ <u>is interpreted to be</u> 0 <u>at</u> $\tilde{\theta}_1 = 0$.

<u>Proof</u>. If $\eta = \infty$, then (4.3.4) is trivially true. So, let $\eta < \infty$. Since $f \in E(n_p, \infty)$, the function $D^{n_p}f$ is analytic and univalent in Δ. Therefore from (3.3.7) for $p \geq 2$ and $2 \leq k \leq n_{p+1}-n_p+1$,

$$(4.3.5) \qquad |a(n_p+k)|^{1/(n_p+k)} \leq \left(A_1 k \frac{d_k \cdots d_2}{d_{n_p+k}\cdots d_2}\right)^{1/(n_p+k)} \times$$
$$\times \prod_{i=2}^{p} \left((n_i-n_{i-1}+1)\, d(n_i-n_{i-1}+1)\cdots d(2)\right)^{1/(n_p+k)}$$

where $A_1 = d(n_1+1)\ldots d(2)|a(n_1+1)|$, is a constant. Now, $(d_k\ldots d_2)^{1/(n_p+k)}$ decreases with k, therefore for $2 \leq k \leq n_{p+1}-n_p+1$

$$(d_k\ldots d_2)^{1/(n_p+k)} \leq (d(n_{p+1}-n_p+1)\ldots d(2))^{1/(n_{p+1}+1)}$$

$$\leq \exp\left(\frac{\log(d(n_{p+1}-n_p+1)\ldots d(2))}{n_{p+1}+1}\right)$$

$$= \exp\left(\frac{\xi_{p+1}-\xi_p}{n_{p+1}+1}\right).$$

Using (3.4.8) and the above inequality in (4.3.5), we obtain for sufficiently large p,

$$\left|\frac{a(n_p+k)}{e(n_p+k)}\right|^{1/(n_p+k)} \leq (1+o(1))\exp\left[\frac{\xi_{p+1}-\xi_p}{n_{p+1}+1} + \frac{\xi_p}{n_p+k} + \frac{p}{n_p}\log\left(1+\frac{n_p}{p}\right)\right].$$

Hence, on proceeding to limits, we get

$$\tau_\Psi(f) = \limsup_{k\to\infty}\left|\frac{a_k}{e_k}\right|^{1/k}$$

$$= \limsup\{|a(n_p+k)|^{1/(n_p+k)} : 2\leq k\leq n_{p+1}-n_p+1 ;\ p\geq 2\}$$

$$\leq \exp\left[\eta\left(2-\frac{\tilde{\theta}_1}{\tilde{\theta}_2}\right) + \tilde{\theta}_1\log\left(1+\frac{1}{\tilde{\theta}_1}\right)\right].$$

The following example shows that if $\eta = \infty$ then, there exists a function F in $E(n_p, \infty)$ having infinite Ψ-type.

<u>Example</u> 4.3.2 Choose a strictly increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers and $\{d_{n_p}\}_{p=1}^{\infty}$ of positive numbers such that

$$\eta = \infty = \lim_{p\to\infty} \xi_p^* .$$

where ξ_p^* and η are defined by (4.3.1) and (4.3.2).

Let

$$a_{j+1} = \begin{cases} \dfrac{\exp(\xi_p)}{2^{p-1}\ d(j+1)\ldots d(2)(j-n_p+1)}, & j = n_p \text{ for some } p. \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$F(z) = \sum_{k=1}^{\infty} a_k z^k .$$

Since, $\limsup_{p\to\infty} \xi_p^* = \infty$, by Theorem (4.3.1), F is entire.

Now, as $\eta = \infty$,

$$\tau_\Psi(F) = \limsup_{k\to\infty} \left|\frac{a_k}{e_k}\right|^{1/k} = \limsup_{p\to\infty} \left|\frac{a(n_p+1)}{e(n_p+1)}\right|^{1/(n_p+1)}$$

$$= \limsup_{p\to\infty} \left[\frac{\exp(\xi_p)}{2^{p-1}}\right]^{1/(n_p+1)}$$

$$= \infty .$$

Thus, F is of infinite Ψ-type. It remains to show that $D^{n_p}F$ is univalent in Δ. In view of (1.2.6), it is enough to prove that

$$\sum_{k=1}^{\infty} (n_{p+k}-n_p+1) \frac{d(n_{p+k}+1)\ldots d(2)}{d(n_{p+k}-n_p+1)\ldots\ d(2)} |a(n_{p+k}+1)|$$

$$\leq d(n_p+1)\ldots d(2)|a(n_p+1)| .$$

Or, equivalently,

$$\sum_{k=1}^{\infty} \frac{\exp(\xi_{p+k}-\xi_p)}{2^k\, d(n_{p+k}-n_p+1)\cdots d(2)} \leq 1. \tag{4.3.6}$$

It is easily seen by inducting upon k that

$$\exp(\xi_{p+k}-\xi_p) = \prod_{i=p+1}^{p+k} (d(n_i-n_{i-1}+1)\cdots d(2))$$
$$\leq d(n_{p+k}-n_p+1)\cdots d(2)$$

and so (4.3.6) is clearly satisfied.

If we further assume that $\limsup_{p\to\infty}(n_p-n_{p-1})<\infty$, in the previous theorem, we get a simpl$\ast$er estimate than in (4.3.4). This is exhibited in the following theorem.

<u>Theorem</u> 4.3.3 <u>Let</u> $f \in E(n_p;\infty)$. <u>Suppose</u> $\limsup_{p\to\infty}(n_p-n_{p-1}) = \mu$; $1 \leqq \mu < \infty$. <u>Then the</u> Ψ-<u>type of</u> f <u>satisfies</u>

$$\tau_{\Psi}(f) \leqq 2(d(\mu+1)\cdots d(2))^{1/\mu}. \tag{4.3.7}$$

<u>Proof</u>. Since the function $D^{n_p}f$ is analytic and univalent in Δ, we have from (3.3.7) for $2 \leq k \leq n_{p+1}-n_p+1$ and sufficiently large p,

$$\left|\frac{a(n_p+k)}{e(n_p+k)}\right|^{1/(n_p+k)} \leq (1+o(1))(d_k\cdots d_2)^{1/(n_p+k)} \times \tag{4.3.8}$$
$$\times \prod_{i=2}^{p}((n_i-n_{i+1}+1)d(n_i-n_{i-1}+1)\cdots d(2))^{1/(n_p+k)}.$$

Let, μ' be such that $\mu < \mu' < \infty$. Then, for sufficiently large p we have $(n_{p+1}-n_p) < \mu'$. Therefore, for $2 \leq k \leq n_{p+1}-n_p+1$,

$$(d_k\cdots d_2)^{1/(n_p+k)} \leq (d(n_{p+1}-n_p+1)\cdots d(2))^{1/n_{p+1}} \tag{4.3.9}$$
$$= 1+o(1), \text{ as } p\to\infty.$$

Using (3.4.8) and the preceeding inequality in (4.3.8), we get for sufficiently large p,

$$(4.3.10)\qquad \left|\frac{a(n_p+k)}{e(n_p+k)}\right|^{1/(n_p+k)} \leq 2(1+o(1)) \prod_{i=2}^{p}(d(n_i-n_{i-1}+1)\ldots d(2))^{1/(n_p+k)}$$

Now, if $a_j \geq o$, $t_j \geq o$, $\Sigma t_j > o$ and $\max\limits_{1\leq j\leq N-1}(\frac{a_j}{j}) \leq \frac{a_N}{N}$, then clearly,

$$(4.3.11)\qquad \frac{\sum_{j=1}^{N} t_j a_j}{\sum_{j=1}^{N} jt_j} \leq \frac{a_N}{N}$$

Further, $\log(d(j+1)\ldots d(2))/j$ is an increasing function of j for $1\leq j\leq \mu$, $\mu = 1,2,\ldots$. Thus, if $1 \leq j \leq \mu$,

$$(4.3.12)\qquad \frac{\log(d(j+1)\ldots d(2))}{j} \leq \frac{\log(d(\mu+1)\ldots d(2))}{\mu}$$

Let $p > p_o$ and $1 \leq \gamma \leq \mu$. Suppose t_γ is the number of j_i's in $[p_o, p]$ such that $n_{j+1}-n_j = \gamma$ for $j = j_i$. Then, by (4.3.11) and (4.3.12)

$$\frac{\sum_{j=p_o+1}^{p} \log(d(n_j-n_{j-1}+1)\ldots d(2)}{\sum_{j=p_o+1}^{p}(n_j-n_{j-1})} = \frac{\sum_{\gamma=1}^{\mu} t_\gamma\cdot\log(d(\gamma+1)\ldots d(2))}{\sum_{\gamma=1}^{\mu}\gamma t_\gamma}$$

$$\leq \frac{\log(d(\mu+1)\ldots d(2))}{\mu}$$

This inequality gives

$$\prod_{i=2}^{p}(d(n_i-n_{i-1}+1)\ldots d(2))^{1/(n_p+k)}$$

$$\leq \exp\left[\frac{\sum_{i=2}^{p}\log(d(n_i-n_{i-1}+1)\ldots d(2))}{n_p}\right]$$

$$\leq \exp\left[o(1)+\frac{\sum_{i=p_o+1}^{p}\log(d(n_i-n_{i-1}+1)\ldots d(2))}{\sum_{i=p_o+1}^{p}(n_i-n_{i-1})}\right]$$

$$\leq \exp\left[o(1)+\frac{\log(d(\mu+1)\ldots d(2))}{\mu}\right].$$

Using the above estimate in (4.3.10) and proceeding to limit

$$\limsup_{k\to\infty}\left|\frac{a_k}{e_k}\right|^{1/k}=\lim\sup\left\{\left|\frac{a(n_p+k)}{e(n_p+k)}\right|^{1/(n_p+k)}: 2\leq k\leq n_{p+1}-n_p+1; p\geq 2\right\}$$

$$\leq 2(d(\mu+1)\ldots d(2))^{1/\mu}.$$

This proves the theorem.

However, in certain conditions Theorem 4.3.2 is applicable while Theorem 4.3.3 can not be applied. This is illustrated by the following example :

<u>Example</u> 4.3.3 Let $c>1$, $M_1 = 10^2$, $M_k = \max\{[k^c], M_{k-1}+1\}$ and $n_1 = 1$. Define a sequence $\{n_p\}_{p=2}^{\infty}$ of positive integers by

$$n_p - n_{p-1} = \begin{cases} 1 & , p \neq M_k \\ [\log\log p], & p = M_k, \ k = 1, 2, \ldots \end{cases}$$

Let $d_n = n^a$, $a \geq 1$ for $n = 1, 2, \ldots$. It is easily seen that $\mu = \lim\limits_{p\to\infty} \sup (n_p - n_{p-1}) = \infty$ so that Theorem 4.3.3 is not applicable for $f \in E(n_p, \infty)$ where the sequence $\{n_p\}_{p=1}^{\infty}$ is defined as above. However, since $\eta = o$ and $\tilde{\theta}_1 = \tilde{\theta}_2 = 1$ Theorem 4.3.2 can be applied and gives that $\tau_{\Psi}(f) \leq 2$.

Theorem 4.3.3 shows that if $(n_p - n_{p-1}) = O(1)$ as $p \to \infty$, then f is of finite Ψ-type. However, in the following example we show that if $\mu = \lim\limits_{p\to\infty} \sup (n_p - n_{p-1}) = \infty$, then there exists an $f \in E(n_p, \infty)$ which is not of finite Ψ-type.

<u>Example</u> 4.3.4 Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $(n_p - n_{p-1}) \geq 2$ for all p and $\lim\limits_{p\to\infty} \sup (n_p - n_{p-1}) = \infty$. Further, we assume that $\{d_n\}_{n=1}^{\infty}$ is an increasing sequence of positive numbers such that

(i) $d_1 = 1$, $\log d(n) \sim \log n$ as $n \to \infty$ (ii) $\zeta_p = o(n_p \log d(n_p))$

(iii) $n_p = o(\zeta_p)$ where

$$\zeta_p = \sum_{i=2}^{p} \log(d(n_i - n_{i-1} + 1) \ldots d(2)).$$

Let Φ be a non-decreasing step function such that

$\Phi(n_1) = \Phi(n_2)$

$$\Phi(n_p) = \frac{\exp(\zeta_p)}{2^{p-1}}, \quad p \geq 2$$

and

$$\Phi(x) = \Phi(n_p), \quad n_p < x \leq n_{p+1}.$$

is univalent in Δ . To this end , in view of (1.2.16) , it is enough to prove that

$$\sum_{k=1}^{\infty} (n_{p+k}-n_p+1) \frac{d(n_{p+k}+1)\cdots d(2)}{d(n_{p+k}-n_p+1)\cdots d(2)} |a(n_{p+k}+1)|$$
$$\leq d(n_p+1)\cdots d(2)|a(n_p+1)| .$$

Or , equivalently to show that

$$\sum_{k=1}^{\infty} \frac{\Phi(n_{p+k})}{d(n_{p+k}-n_p+1)\cdots d(2)} \leq \Phi(n_p) .$$

Using the definition of Φ , this inequality reads as

(4.3.13) $$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\exp(\zeta_{p+k}-\zeta_p)}{d(n_{p+k}-n_p+1)} \leq 1 .$$

Since , an induction argument on k gives for k = 1, 2, ...

$$\exp(\zeta_{p+k}-\zeta_p) = \prod_{p+1}^{p+k} d(n_i-n_{i-1}+1)\cdots d(2)$$
$$\leq d(n_{p+k}-n_p+1)\cdots d(2) .$$

the inequality in (4.3.13) is clearly true.

<u>Remarks</u> 1. In Theorem 4.3.3 , it is sufficient to take the function f in the class $E(n_p, R)$ if the sequence $\{d_n\}_{n=1}^{\infty}$ of positive number satisfies the condition

(4.3.14) $$\lim_{p\to\infty} (d_{n_p}\cdots d_1)^{1/n_p} = \infty .$$

In fact , if $f \in E(n_p, R)$, $\limsup_{p\to\infty}(n_p-n_{p-1}) = \mu < \infty$ and (4.3.14) holds , then f is necessarily entire so that $f \in E(n_p, \infty)$. To see this , from (4.3.10) , we have for

sufficiently large p

$$(4.3.15)\quad |a(n_p+k)|^{1/(n_p+k)}$$

$$\leq 2(1+o(1))\exp\Big[\frac{1}{n_p}\sum_{i=2}^{p}\log(d(n_i-n_{i-1}+1)\ldots d(2)) - \frac{1}{(n_p+k)}\sum_{i=2}^{n_p+k}\log d(i)\Big].$$

But, since $(n_r-n_{r-1})\leq \mu'$, $\mu<\mu'<\infty$, for sufficiently large p, we get

$$\frac{1}{n_p}\sum_{i=2}^{p}\log(d(n_i-n_{i-1}+1)\ldots d(2)) = O(1)$$

Thus, using (4.3.14), the inequality (4.3.15), gives

$$\limsup_{k\to\infty}|a_k|^{1/k} = \lim\sup\{|a(n_p+k)|^{1/(n_p+k)} ; 2\leq k\leq n_{p+1}-n_p+1; p\geq 2\}$$
$$= 0.$$

This proves that f is an entire function implying thereby $f \in E(n_p,\infty)$.

2. The inequality (4.3.8) in Theorem 4.3.3 can be improved by imposing suitable additional restriction on the sequence $\{d_n\}_{n=1}^{\infty}$. For example, let the sequence $\{d_n\}_{n=1}^{\infty}$ be such that

$$(4.3.16)\qquad \frac{(d_{n+2})^n}{d_{n+1}\,d_n\ldots d_2} \geq \frac{2}{3}(n+1),\quad n=1,2,\ldots$$

Dur to (4.3.16), the function $\tilde{s}(j)$ defined by

$$\tilde{s}(j) = \frac{\log(d(j+1)\ldots d(2)) + \log(j+1)}{j}$$

is an increasing function of j and so, for $j = 1, 2, \ldots, \mu$; $\mu = 1, 2, \ldots$

$$(4.3.17)\quad \frac{\log(d(j+1)\ldots d(2))+\log(j+1)}{j} \leq \frac{\log(d(\mu+1)\ldots d(2))+\log(\mu+1)}{\mu}$$

Let t_γ be the same as in the proof of Theorem 4.3.3. Then, by using (4.3.17) and (4.3.11), we get

$$\frac{\sum_{j=p_o+1}^{p} (\log(d(j+1)\ldots d(2))+\log(j+1)}{\sum_{j=p_o+1}^{p} (n_j-n_{j-1})}$$

$$= \frac{\sum_{\gamma=1}^{\mu} t_\gamma \,(\log(d(\gamma+1)\ldots d(2))+\log(\gamma+1))}{\sum_{\gamma=1}^{\mu} \gamma t_\gamma}$$

$$\leq \frac{\log(d(\mu+1)\ldots d(2)+\log(\mu+1)}{\mu}.$$

Thus,

$$(4.3.18)\quad \prod_{i=2}^{p}((n_i-n_{i-1}+1)d(n_i-n_{i-1}+1)\ldots d(2))^{1/(n_p+k)}$$

$$\leq \exp[o(1)+\frac{\sum_{i=p_o+1}^{p} \log(d(n_i-n_{i-1}+1)\ldots d(2))+\log(n_i-n_{i-1}+1)}{\sum_{i=p_o+1}^{p} (n_i-n_{i-1})}]$$

$$\leq \exp[o(1) + \frac{\log(d(\mu+1)\ldots d(2) + \log(\mu+1)}{\mu}].$$

Now , by (4.3.8) and (4.3.9) , we have

$$\left|\frac{a(n_p+k)}{e(n_p+k)}\right|^{1/(n_p+k)} \leq (1+o(1)) \left(\prod_{i=2}^{p} ((n_i-n_{i-1}+1)d(n_i-n_{i-1}+1)\ldots d(2))\right)^{1/(n_p+k)}$$

Therefore , by using (4.3.18) in this inequality and proceeding to limit , we get

$$\tau_{\Psi}(f) \leq (\mu+1)^{1/\mu} (d(\mu+1)\ldots d(2))^{1/\mu}.$$

which is clearly an improvement of (4.3.7). Note that (4.3.16) is satisfied for $d_n = n^a$, $a \geq 1$, $d_n = e^n$ etc.

3. If $n_o = o$ and $(n_p-n_{p-1}) = 1$ for $p \geq 1$, then Theorem 4.3.3 gives that $\tau_{\Psi}(f) \leq 2d_2$, a result proved by Juneja and Shah [41] . Infact , our Theorem 4.3.3 shows that the bound $\tau_{\Psi}(f) \leq 2d_2$ continues to hold even under the weaker condition $\limsup_{p\to\infty} (n_p-n_{p-1}) = 1$. For $d_n \equiv n$, a result of Shah [80] follows from Theorem 4.3.3.

4.4 In this section , we study the Ψ-order (c.f.Section 4.1) of an entire function in relation to univalent functions with univalent Gelfond-Leontev derivatives . First we find a characterization formula for Ψ-order of an entire function interms of the coefficients of its power series. Using this characterization,an upper bound for Ψ-order of an entire function f is obtained when some of the Gelfond-Leontev derivatives of f are assumed to be analytic and univalent in the unit disc Δ .

We prove

<u>Theorem</u> 4.4.1 <u>Let</u> $f(z) = \sum_{n=0}^{\infty} a_n z^n$ <u>be an entire function of</u> Ψ-<u>order</u> $\rho_\Psi(f)$. <u>Then</u>

$$\rho_\Psi(f) = \limsup_{n\to\infty} \frac{\log e_n}{\log |a_n|} . \tag{4.4.1}$$

we need the following lemma :

<u>Lemma</u> 4.4.1 <u>Let</u> Ψ <u>be defined by</u> (1.3.14) . <u>Let</u> $\gamma_n = \min_{x>0} \Psi(x^a)x^{-n}$, $n > 0$. <u>Then</u> ,

$$\gamma_n \leq e_n d_n^{n(1-\frac{1}{a})} \frac{e(n+a)}{a} \tag{4.4.2}$$

<u>Proof</u>. Since $\{d_n\}_{n=1}^{\infty}$ is non-decreasing , we note that for any pair of positive integers k and n , $e_k \leq e_n d_n^{n-k}$. Thus ,

$$\Psi(x^a) = \sum_{k=0}^{\infty} e_k x^{ak} \leq e_n d_n^n \sum_{k=0}^{\infty} d_n^{-k} x^{ak}$$

Let $0 < w < 1$. Setting $x_w = wd_n^{1/n}$, we get

$$\Psi(x_w^a)x_w^{-n} \leq e_n d_n^{n(1-\frac{1}{a})} \frac{w^{-n}}{(1-w^a)}$$

Choosing $w = (n/(n+a))^{1/a}$ to minimize the right side of the above inequality , we have

$$\gamma_n \leq \min_{0<w<1} \Psi(x_w^a)x_w^{-n}$$

$$\leq e_n d_n^{n(1-\frac{1}{a})} \frac{e(n+a)}{a} .$$

<u>Proof of Theorem</u> 4.4.1 By Cauchy's inequality , we get for $|z| = r$

$$|a_n| \leq M(r)r^{-n}$$

where $M(r) = \max_{|z|=r} |f(z)|$. Since f is of Ψ-order $\rho_\Psi(f) \cong \rho$, for given $\varepsilon > 0$

$$|f(re^{i\theta})| \leq K_1 \Psi(r^{\rho+\varepsilon})$$

where K_1 is a constant . Then , for $|z| = r$ ·

$$|a_n| \leq K_1 \ \Psi(r^{\rho+\varepsilon}) r^{-n}$$

Now , by using Lemma 4.4.1 in this inequality , we obtain

$$|a_n| \leq K_1 \ e_n \ d_n^{n(1-\frac{1}{\rho+\varepsilon})} \frac{e(n+\rho+\varepsilon)}{\rho+\varepsilon}$$

Since $d_n \to \infty$ as $n \to \infty$, this gives on proceeding to limit

$$(4.4.3) \qquad \limsup_{n\to\infty} \frac{\log e_n}{\log|a_n|} \leq \rho.$$

To show that equality holds in (4.4.1) , assume that

$$\limsup_{n\to\infty} \frac{\log e_n}{\log |a_n|} < \rho$$

Then, there exists a positive number $\rho_1 < \rho$ and a positive integer n_0 such that $|a_n| < e_n^{1/\rho_1}$ for $n > n_0$. Thus , it follows that for $|z| = r$

$$|f(z)| \leq \sum_{n=0}^{n_0} |a_n| r^n + \sum_{n=n_0+1}^{\infty} |a_n| r^n$$
$$< O(1) + \sum_{n=n_0+1}^{\infty} e_n^{1/\rho_1} r^n$$

Choose $N(r) = \log \Psi(r^{\rho_1})/\log r$. It is easily seen that $N(r) \to \infty$ as $r \to \infty$. Since for all values of k and n , $e_n < e_k \, d_k^{k-n}$, we have

$$\sum_{n=o}^{\infty} e_n^{1/\rho_1} r^n < \sum_{n=o}^{\infty} e_k^{1/\rho_1} d_k^{(k-n)/\rho_1} r^n$$

$$= d_k^{k/\rho_1} e_k^{1/\rho_1} \sum_{n=o}^{\infty} \left(\frac{r}{d_k^{1/\rho_1}}\right)^n ,$$

Let k be chosen such that $(r/d_k^{1/\rho_1}) < 1$. Then

$$\sum_{n=o}^{\infty} e_n^{1/\rho_1} r^n < \frac{d_k^{(k+1)\rho_1} e_k^{1/\rho_1}}{(d_k^{1/\rho_1} - r)} . \tag{4.4.4}$$

Since the left side of (4.4.4) is independent of k, letting $k \to \infty$, we get

$$\sum_{n=o}^{\infty} e_n^{1/\rho_1} r^n < 1 .$$

Thus , as $n \to \infty$,

$$\sum_{n=N(r)}^{\infty} e_n^{1/\rho_1} r^n = o(1) .$$

Now , $r^{N(r)} = \exp(N(r) \log r) = \Psi(r^{\rho_1})$. Hence from (4.4.3) we have

$$|f(z)| \leq O(1) + \sum_{n_o+1}^{N(r)} e_n^{1/\rho_1} r^n + o(1)$$

$$\leq O(1)\ \Psi(r^{\rho_1}) .$$

Since $\rho_1 < \rho$ and ρ is the Ψ-order of f , the above inequality contradicts the definition of Ψ-order of f. Thus , equality must hold in (4.4.3). This proves the theorem .

<u>Corollary</u> 4.4.1 <u>Let</u> $f(z) = \sum_{n=o}^{\infty} a_n z^n$ <u>be an entire function of Ψ-order $\rho_\Psi(f)$. Let the sequence $\{d_n\}_{n=1}^{\infty}$ in the definition of Ψ be such that</u> log d(n) <u>is the restriction of a slowly oscillating function on positive integers. Then ,</u>

$$\rho_\Psi(f) = \limsup_{n \to \infty} \frac{n \log d(n)}{-\log|a_n|} . \tag{4.4.5}$$

<u>Proof</u>. As $\log d(n)$ is the restriction of a slowly oscillating function on positive integers we have by (3.4.8), as $n \to \infty$

$$\log (1/e_n) = \sum_{i=1}^{n} \log d(i) \sim n \log d(n).$$

Thus, by the above Theorem, we get (4.4.5).

<u>Remark</u>. For $d_n \equiv n$, (4.4.5) gives the coefficient characterization for the classical order (c.f. Section 1.3).

<u>Theorem</u> 4.4.2 <u>Let</u> $f \in E(n_p, \infty)$ <u>have</u> Ψ-<u>order</u> $\rho_\Psi(f)$. <u>Let</u> $n_p \sim n_{p+1}$ as $p \to \infty$. <u>If</u> $\log d(n)$ <u>is the restriction of a slowly oscillating function on positive integers</u>, <u>then the</u> Ψ-<u>order of</u> f <u>satisfies</u>

$$\rho_\Psi(f) \leq \frac{1}{1 - \limsup\limits_{p \to \infty} \dfrac{\log d(n_p - n_{p-1})}{\log d(n_p)}} \tag{4.4.6}$$

<u>Proof</u>. Since $f \in E(n_p, \infty)$, $D^{n_p} f$ is analytic and univalent in Δ, we have (3.3.7) for $2 \leq k \leq n_{p+1} - n_p + 1$ and for sufficiently large p,

$$|a(n_p+k)|^{1(n_p+k)} \leq (1+o(1)) \left(\frac{d_k \cdots d_2}{d_{n_p+k} \cdots d_2}\right)^{1/(n_p+k)} \times \tag{4.4.7}$$

$$\times \prod_{i=2}^{p} ((n_i - n_{i-1}+1) d(n_i - n_{i-1}+1) \cdots d(2))^{1/(n_p+k)}.$$

Further, we have

$$(d_k \cdots d_2)^{1/(n_p+k)} \leq (d(n_{p+1} - n_p + 1) \cdots d(2))^{1/(n_{p+1}+1)}$$

and

$$(d_{n_p+k}\cdots d_2)^{-1/(n_p+k)} \leq (d(n_p+2)\cdots d(2))^{-1/(n_p+2)} .$$

Using these inequalities and (3.4.8) in (4.4.7), it follows that for sufficiently large p and $2 \leqq k \leqq n_{p+1}-n_p+1$,

$$(4.4.8) \quad |a(n_p+k)|^{1/(n_p+k)} \leq \frac{2(1+o(1))}{(d(n_p)\cdots d(2))^{1/n_p}} \times$$

$$\times \prod_{i=2}^{p+1} (d(n_i-n_{i-1}+1))^{(n_i-n_{i-1})/n_p} .$$

Let,

$$M_p = \max\{\log d(n_i-n_{i-1}+1) : 2 \leq i \leq p\}$$

As log d(n) is the restriction of a S.O. function on positive integers, by using (3.4.8) we deduce that

$$\log \left[\frac{\prod_{i=2}^{p+1} (d(n_i-n_{i-1}+1))^{(n_i-n_{i-1})/n_p}}{(d(n_p)\cdots d(2))^{1/n_p}} \right]$$

$$= \frac{1}{n_p}\left[\sum_{i=2}^{p+1}(n_i-n_{i-1})\log d(n_i-n_{i-1}+1) - \sum_{i=1}^{n_p} \log d(i)\right]$$

$$\leq \frac{n_{p+1}}{n_p} M_{p+1} - \log d(n_p) .$$

Consequently, for sufficiently large p, and $2 \leq k \leq n_{p+1}-n_p+1$,

$$\frac{(n_p+k)\log d(n_p+k)}{-\log|a(n_p+k)|} \leq \frac{\log d(n_{p+1}+1)}{\log d(n_p) - \frac{n_{p+1}}{n_p} M_p - \log 2}$$

Since $n_p \sim n_{p+1}$ as $p \to \infty$ and $\log d(n)$ is the restriction of a slowly oscillating function on positive integers , we have $\log d(n_p) \sim \log d(n_{p+1})$ as $p \to \infty$, we get on proceeding to limit in the above inequality

$$(4.4.9) \qquad \rho_{\psi}(f) \leq \frac{1}{1-\limsup\limits_{p\to\infty} \frac{M_p}{\log d(n_p)}} .$$

If M_p is bounded , there is nothing to prove . So , let $M_p \to \infty$ as $p \to \infty$.

For $p \geq 2$, let

$$A_p = \frac{\log d(n_p - n_{p-1} + 1)}{\log d(n_p)}$$

and

$$B_p = \frac{M_p}{\log d(n_p)}$$

But as $M_p \to \infty$, for each $p \geq 2$ there is some q_p ; $q_p \leq p$, such that $M_p = \log d(n_{q_p} - n_{q_p - 1} + 1)$. Hence $B_p \leq A_{q_p}$. Letting $q_p \to \infty$, we get

$$\limsup_{p\to\infty} B_p \leq \limsup_{p\to\infty} A_p .$$

Now on applying the above inequality , the estimate in (4.4.6) follows from (4.4.9). This completes the proof of Theorem 4.4.2.

The following corollary shows that the hypothesis of Theorem 4.4.2 can be weakened if, in particular , $d_n \equiv n^a$ some real $a > 0$. For an entire function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, denote

$$D_1^{n_p} f(z) = \sum_{k=0}^{\infty} ((n_p+k)\ldots(k+1))^a \, a(n_p+k) z^k .$$

Note that $D_1^{n_p} f(z) = D^{n_p} f(z)$ if $d_n \equiv n^a, a > 0$

Corollary 4.4.2 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of Ψ-order $\rho_\psi(f)$. Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $D_1^{n_p} f$ is univalent in Δ. If $\log n_p \sim \log n_{p+1}$, then

$$(4.4.10) \qquad \rho_\psi(f) \leq \frac{1}{1-\limsup\limits_{p\to\infty} \dfrac{\log(n_p-n_{p-1})}{\log n_p}} .$$

Proof. Since $D_1^{n_p} f$ is univalent in the unit disc Δ, the inequality in (4.4.7) reduces to, for $2 \leq k \leq n_{p+1}-n_p+1$ and large p,

$$|a(n_p+k)| \leq (1+o(1)) \left(\frac{k!}{(n_p+k)!}\right)^a \prod_{i=2}^{p} (n_i-n_{i-1}+1)((n_i-n_{i-1}+1)!)^a$$

This inequality is of the form (3.4.7). Therefore, using the same technique as applied to obtain (3.4.10) from (3.4.7), we get

$$|a(n_p+k)|^{1/(n_p+k)} \leq 2\left[\frac{A_1 k_2^p e^{n_1+1-p}}{k_1} \left(\frac{k^3}{n_p+k}\right)^{\frac{1}{2}}\right]^{a/(n_p+k)} \times$$

$$\times \max\left\{ \frac{4^{1/n_p}}{n_p} \prod_{i=2}^{p} (n_i-n_{i-1}+1)^{(n_i-n_{i-1})/n_p}, \right.$$

$$\left. \frac{n_{p+1}^{1/n_{p+1}}}{n_{p+1}} \prod_{i=2}^{p+1} (n_i-n_{i-1}+1)^{(n_i-n_{i-1})/n_{p+1}} \right\}^a$$

Thus , for sufficiently large p ,

$$(4.4.11) \quad |a(n_p+k)|^{1/(n_p+k)} \leq K_1 \frac{1}{n_p^a} \prod_{i=2}^{p} (n_i-n_{i-1}+1)^{a(n_i-n_{i-1})/n_p}$$

where K_1 is a constant . Let

$$M_p = \max \{\log(n_i-n_{i-1}+1) : 2 \leq i \leq p\} .$$

Then ,

$$\log\left[\frac{\prod_{i=2}^{p} (n_i-n_{i-1}+1)^{a(n_i-n_{i-1})/n_p}}{n_p^a} \right]$$

$$= \frac{a}{n_p} \left[\sum_{i=2}^{p} (n_i-n_{i-1})\log(n_i-n_{i-1}+1) - n_p \log n_p \right]$$

$$\leq a(M_p - \log n_p) .$$

Consequently , by (4.4.11) for sufficiently large p ,

$$\frac{(n_p+k)\log(n_p+k)}{-\log|a(n_p+k)|} \leq \frac{\log(n_{p+1}+1)}{\log n_p - M_p - \log K_1^{1/a}} .$$

Since , $\log n_p \sim \log n_{p+1}$ as $p \to \infty$, we get on proceeding to limit in the above inequality

$$(4.4.12) \qquad \rho_\psi(f) \leq \frac{1}{1 - \limsup_{p\to\infty} \frac{M_p}{\log n_p}}$$

If M_p is bounded , (4.4.10) is trivially true. So , let $M_p \to \infty$ as $p \to \infty$. Then , for each $p \geq 2$ let

$$A'_p = \frac{\log(n_p-n_{p-1}+1)}{\log n_p} \quad \text{and} \quad B'_p = \frac{M_p}{\log n_p}$$

Since $M_p \to \infty$ as $p \to \infty$, for each $p \geq 2$, there is some q_p; $q_p \leq p$ such that $M_p = \log(n_{q_p} - n_{q_p - 1} + 1)$. Hence $B'_p \leq A'_{q_p}$. Letting $q_p \to \infty$, we obtain

$$\limsup_{p \to \infty} B'_p \leq \limsup_{p \to \infty} A'_p \; .$$

Thus, by (4.4.12) and the above inequality,

$$\rho_\Psi(f) \leq \frac{1}{1 - \limsup\limits_{p \to \infty} \dfrac{\log(n_p - n_{p-1})}{\log n_p}} \; .$$

This completes the proof of the corollary.

<u>Corollary</u> 4.4.3 <u>Assume the hypothesis of Theorem</u> 4.4.2. <u>If</u> $\log d(n_p - n_{p-1}) = o(\log d(n_p))$ <u>as</u> $p \to \infty$ <u>then</u> $\rho_\Psi(f) \leq 1$.

<u>Remark</u>. For $d_n \equiv n^a$, $a > 0$, we get from Corollary 4.4.1 and (1.3.8) that $a\Lambda = \rho_\Psi(f)$, where Λ is the classical order of f defined in Section 1.3. Thus, the inequality in (4.4.10) in this case reduces to

$$\Lambda \leq \frac{1}{a\left(1 - \limsup\limits_{p \to \infty} \dfrac{\log(n_p - n_{p-1})}{\log n_p}\right)} \qquad (4.4.13)$$

For $a = 1$, (4.4.13) is due to Shah and Trimble [86].

We observe that if $0 \leq \rho_\Psi(f) \leq 1$, then Theorem 4.4.2 gives essentially no relationship between $D^{n_p} f$ and the Ψ-order $\rho_\Psi(f)$ of an entire function f. Infact, no such relation of this nature exists is illustrated in the following Theorem.

<u>Theorem</u> 4.4.3 <u>Let</u> $o \leq \rho \leq 1$. <u>Let</u> $\{n_p\}_{p=1}^{\infty}$ <u>be a strictly increasing sequence of positive integers</u> . <u>Then</u> , <u>there is an entire function</u> h <u>of finite</u> Ψ-<u>order</u> $\rho_\Psi(h) \equiv \rho$ <u>such that</u> $D^n h$ <u>is analytic and univalent in</u> Δ , <u>if and only if</u> , $n = n_p$ <u>for some</u> p.

<u>Proof</u>. Suppose $\rho > o$ and $\{d_n\}_{n=1}^{\infty}$ is an increasing sequence of positive numbers such that log d(n) is the restriction of a slowly oscillating (c.f. section 3.4) function on positive integers and $d_n \to \infty$. Let

$$h_{j+1} = \begin{cases} \dfrac{1}{2^p (d(j+1)\ldots d(2))^{1/\rho (j-n_p+1)}} , & j = n_p \text{ for some } p \\ o & , \text{otherwise.} \end{cases}$$

Define,

$$h(z) = \sum_{j=1}^{\infty} h_j z^j .$$

Then , h(z) is an entire function and

$$\begin{aligned} \rho_\Psi(h) &= \limsup_{k\to\infty} \frac{k \log d(k)}{-\log |h_k|} \\ &= \limsup_{p\to\infty} \frac{(n_p+1) \log d(n_{p+1}+1)}{p \log 2 + \frac{1}{\rho} \log(d(n_p+1)\ldots d(2))} \\ &= \rho . \end{aligned}$$

To show that $D^{n_p} h$ given by

$$D^{n_p} h(z) = \sum_{k=1}^{\infty} \frac{d(n_{p+k}+1)\ldots d(2)}{d(n_{p+k}-n_p+1)\ldots d(2)} \; h(n_{p+k}+1) z^{n_{p+k}-n_p+1}$$

is univalent in Δ , in view of (1.2.16) , it is enough to prove

$$(4.4.14) \quad \sum_{k=1}^{\infty} (n_{p+k}-n_p+1) \frac{d(n_{p+k}+1)\ldots d(2)}{d(n_{p+k}-n_p+1)\ldots d(2)} h(n_{p+k}+1)$$

$$\leq d(n_p+1)\ldots d(2) \quad h(n_p+1) \ .$$

Now , since $\rho \leq 1$, we have

$$\sum_{k=1}^{\infty} (n_{p+k}-n_p+1) \frac{d(n_{p+k}+1)\ldots d(2)}{d(n_{p+k}-n_p+1)\ldots d(2)} h(n_{p+k}+1)$$

$$= \frac{1}{2^p} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{(d(n_{p+k}+1)\ldots d(2))^{1-\frac{1}{\rho}}}{d(n_{p+k}-n_p+1)\ldots d(2)}$$

$$< \frac{(d(n_p+1)\ldots d(2))^{1-\frac{1}{\rho}}}{2^p}$$

and so (4.4.14) follows. Further , since $D^{n+1}h(o) = o$ unless $n = n_p$ for some p , only $D^{n_p}h$ are univalent in Δ .

If $\rho = o$, we take h^*_{j+1} defined by

$$h^*_{j+1} = \begin{cases} \dfrac{1}{2^{p+(d(j+1)\ldots d(2))}(j-n_p+1)} & , \ j = n_p \text{ for some } p \\ o & , \text{ otherwise} \end{cases}$$

in place of h_{j+1} in the power series expansion of the function $h(z)$.

Our next theorems shows that with suitable growth condition on the sequence $\{n_p\}_{p=1}^{\infty}$, it is possible to construct an entire function F with a given Ψ-order $\rho_\Psi(F) \geq 1$ and having $D^{n_p}f$ univalent in Δ for $p \geq P$, where P is a natural number.

Theorem 4.4.4 Let $1 \leq \rho < \infty$. Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of non-negative integer such that $\tilde{\beta} = \liminf_{p \to \infty} \frac{n_{p+1}}{n_p} > \rho$. Then, there is a natural number P and an entire function f of Ψ-order $\rho_\Psi(f) = \rho$ such that $D^n f$ is univalent in Δ if and only only if $n = n_p$ for some p, where $p \in \{P, P+1, \ldots\}$.

Proof. Let $\tilde{\beta} = \frac{b+\rho}{2} \geq 2$. Let p_1 be a positive integer such that $n_{p+1}/n_p \geq \tilde{\beta}$ for $p \geq p_1$. Let $P \geq p_1$ be such that for $p \geq P$,

$$(4.4.15) \quad (d(n_{p+1}+1)\ldots d(2))^{1-\frac{1}{\rho}} \leq (d(n_p+1)\ldots d(2))^{1-\frac{1}{\rho}} d(n_{p+1}-n_p+1)\ldots d(2).$$

Let,

$$a_{j+1} = \begin{cases} \dfrac{1}{2^p (d(j+1)\ldots d(2))^{1/\rho}(j-n_p+1)}, & j=n_p \text{ some } p \geq P. \\ 0, & \text{otherwise.} \end{cases}$$

where, $\{d_n\}_{n=1}^{\infty}$ is a increasing sequence of positive numbers such that $d_n \to \infty$ as $n \to \infty$ and $\log d(n)$ is the restriction of a slowly oscillating function on positive integers.

Define,

$$F(z) = \sum_{j=1}^{\infty} a_j z^j .$$

Then , it is easily seen that F is an entire function of Ψ-order $\rho_\Psi(F) \equiv \rho$. It remains to show that each $D^{n_p}F$ is univalent in Δ and , in view of (1.2.16) , it is enough to prove that , for $p \geq P$,

$$\sum_{k=1}^{\infty} (n_{p+k}-n_p+1) \frac{d(n_{p+k}+1)\ldots d(2)}{d(n_{p+k}-n_p+1)\ldots d(2)} |a(n_{p+k}+1)|$$

$$\leq d(n_p+1)\ldots d(2)|a(n_p+1)| .$$

Using (4.4.15) and following the lines of proof of (4.4.14) the above inequality is easily seen to be true.

CHAPTER V

G-L ABSOLUTE STARLIKE AND G-L ABSOLUTE CONVEX FUNCTIONS

5.1 The present chapter is devoted to the study of functions f analytic in the unit disc $\Delta = \{z : |z| < 1\}$, such that its Gelfond-Leontev derivatives $D^k f$, $k = 0, 1, 2, \ldots$, given by (1.4.3), satisfy certain infinite system of inequalities. We first define two subclasses of the class of functions that are analytic and univalent in the unit disc Δ.

<u>Definition</u> 5.1.1 A function f, analytic in Δ, is said to be <u>G-L</u> <u>absolute</u> <u>starlike</u> if $D^k f$, $k = 1, 2, \ldots$, are analytic in Δ and the following system of inequalities are satisfied

$$(5.1.1) \qquad \sum_{n=2}^{\infty} n \frac{|D^{n+k} f(0)|}{d_1 d_2 \cdots d_n} \leq \frac{|D^{k+1} f(0)|}{d_1}, \quad k = 0, 1, 2, \ldots$$

It follows from (5.1.1) that for a G-L absolute starlike function f and every non-negative integer k, the image of Δ under $D^k f$ is starlike with respect to the point $D^k f(0)$. Also, it is readily seen that for a G-L absolute starlike function f, its Gelfond-Leontev derivatives $D^k f$, $k = 0, 1, 2, \ldots$, are analytic and univalent in Δ. This implies that f is of finite Ψ-type [41]. By considering the function $F(z) = z + \sum_{n=2}^{\infty} a_n z^n$, where $a_n = (1 - 1/\sqrt{2})^n$ and taking $d_n \equiv 1$, we observe that, in general, G-L absolute starlike functions need not be entire.

However, if $(d_n \cdots d_1)^{1/n} \to \infty$ as $n \to \infty$, then f becomes necessarily entire (c.f. Section 4.3).

We shall denote by $A_s(D)$, the class of all G-L absolute starlike functions f for which $D^k f$, $k = 0, 1, 2, \ldots$, are entire and satisfy the normalization $Df(0) - d_1 = f(0) = 0$. Let $B_s(D)$ denote the subclass of $A_s(D)$ consisting of those functions f for which $D^k f(0)$, $k = 2, 3, \ldots$, are real and non-negative.

<u>Definition</u> 5.1.2 A function f, analytic in Δ, is said to be <u>G-L absolute convex if</u> $D^k f$, $k = 1, 2, \ldots$, are analytic in Δ and the following system of inequalities are satisfied

$$(5.1.2) \qquad \sum_{n=2}^{\infty} n^2 \frac{|D^{n+k} f(0)|}{d_1 d_2 \cdots d_n} \leq \frac{|D^{k+1} f(0)|}{d_1}, \quad k = 0, 1, 2, \ldots$$

For a G-L absolute convex function f, the image $D^k f(\Delta)$ is convex for every non-negative integer k. Further, (5.1.2) shows that for such functions, f and its Gelfond-Leontev derivative $D^k f$, $k = 0, 1, 2, \ldots$, are convex univalent in Δ.

Let $A_c(D)$ denote the class of all G-L absolute convex functions for which $D^k f$, $k = 0, 1, 2, \ldots$, are entire and satisfy the normalization $Df(0) - d_1 = f(0) = 0$. Likewise, let $B_c(D)$ denote the subclass of $A_c(D)$ for which $D^k f(0)$, $k = 2, 3, \ldots$, are real and non-negative.

Obviously, $A_c(D) \subset A_s(D)$. For $d_n \equiv n$, the class $A_s(D)$ and $A_c(D)$ reduce respectively to the classes of absolute starlike and absolute convex functions introduced and studied by Buckholtz and Shah [17].

Let

$$\Psi(z) = \sum_{n=0}^{\infty} e_n z^n \tag{5.1.3}$$

where $e_0 = 1$, $e_n = (d_1 d_2 \cdots d_n)^{-1}$ and $e_{n+1}/e_n \to 0$ as $n \to \infty$. Clearly , $\Psi(z)$ is an entire function . Let $\hat{\beta}$ be the real root of the equation

$$\Psi'(w) - 2e_1 = 0. \tag{5.1.4}$$

Then , it is easily seen that the equation (5.1.4) has no other root in $|w| \leq \hat{\beta}$.

Similarly , let $\hat{\alpha}$ be the real root of the equation

$$w\Psi''(w) + \Psi'(w) - 2e_1 = 0 \tag{5.1.5}$$

Again , it is seen that the equation (5.1.5) has no other root in $|w| \leq \hat{\alpha}$.

In Section 5.2 of this chapter , we derive some preliminary results concerning the generalization of Appell polynomials that are needed in the subsequent sections. In Section 5.3 , we find sharp coefficient bounds for functions belonging to the classes $A_s(D)$ and $A_c(D)$. Two polynomial representations are developed in the process for entire functions of finite Ψ-type . Further , we show that certain entire functions can be decomposed into sum of two G-L absolute starlike or two G-L absolute convex functions. Finally , in Section 5.4 , a simplicial representation formula

for functions belonging to the classes $B_s(D)$ and $B_c(D)$ are obtained . For $d_n \equiv n$, some of the results of Buckholtz and Shah [17] follow from our results .

5.2 Let U_k , $k = 1, 2, \ldots$ be the linear functional defined on $A_s(D)$ by

$$(5.2.1) \qquad U_k(f) = 2\,\frac{D^k f(o)}{d_1} - \left[\frac{d}{dz}(D^{k-1} f(z))\right]_{z=1} .$$

Since , for functions f in $B_s(D)$, $D^k f(o)$, $k = 2, 3, \ldots$ are real and non-negative , (5.1.1) may be written as

$$\sum_{n=2}^{\infty} n\,\frac{D^{n+k} f(o)}{d_1 d_2 \cdots d_n} = \left[\frac{d}{dz}(D^k f(z))\right]_{z=1} - \frac{D^{k+1} f(o)}{d_1}$$
$$\leq \frac{D^{k+1} f(o)}{d_1} , \quad k = o, 1, 2, \ldots$$

or

$$(5.2.2) \qquad 2\,\frac{D^{k+1} f(o)}{d_1} - \left[\frac{d}{dz}(D^k f(z))\right]_{z=1} \geq o , \quad k = o, 1, 2, \ldots$$

Thus , in view of (5.2.1) and (5.2.2) ,

$$(5.2.3) \qquad U_{k+1}(f) \geq o , \quad k = o, 1, 2, \ldots$$

if and only if , $f \in B_s(D)$.

For the function Ψ , defined by (5.1.3) , we define the polynomial sequence $\{p_n(z)\}_{n=o}^{\infty}$ by the relation

$$(5.2.4) \qquad \sum_{n=o}^{\infty} p_n(z) w^n = \Psi(zw)\left(\frac{2}{d_1} - \Psi'(w)\right)^{-1} .$$

The function $\Psi(zw)(\frac{2}{d_1} - \Psi'(w))^{-1}$ is called the generating function for the polynomial sequence $\{p_n(z)\}_{n=o}^{\infty}$. The polynomials $p_n(z)$ are generalization of Appell polynomials $\hat{p}_n(z)$ (c.f. Section 1. 4). To study the properties of the sequence $\{p_n(z)\}_{n=o}^{\infty}$, we define the sequence $\{u_n\}_{n=o}^{\infty}$ by

(5.2.5) $$\sum_{n=o}^{\infty} u_n w^n = (\frac{2}{d_1} - \Psi'(w))^{-1} .$$

where "," denotes the derivative with respect to w. From (5.2.5), we deduce the recurrence formula

(5.2.6) $$u_o = d_1 , \quad u_n = \sum_{k=o}^{n-1} (n-k+1) \frac{u_k}{d_2 \cdots d_{n-k+1}} , \quad n = 1, 2, \ldots$$

so that $u_n > o$ for all n. It is easily seen from (5.2.5) and (5.2.4) that

(5.2.7) $$p_n(o) = u_n , \quad p_n(z) = \sum_{k=o}^{n} u_{n-k} e_k z^k.$$

Thus, the Gelfond-Leontev derivative Dp_{n+1} of the polynomial $p_n(z)$ satisfies the relation

$$Dp_{n+1}(z) = p_n(z).$$

We now define the normalized polynomial $P_n(z)$ by

(5.2.8) $$P_n(z) = \frac{p_n(z) - p_n(o)}{e_1 \ Dp_n(o)}$$

$$= \frac{p_n(z) - p_n(o)}{e_1 \ u_{n-1}}$$

$$= \sum_{k=1}^{n} \frac{u_{n-k}}{u_{n-1}} \frac{e_k}{e_1} z^k .$$

Let V_k , $k = 1, 2, \ldots$ be the linear functional defined on $A_c(D)$ by

$$(5.2.9) \qquad V_k(f) = \frac{2D^k f(o)}{d_1} - \left[\frac{d}{dz}(D^{k-1}f(z))\right]_{z=1} - \left[\frac{d^2}{dz^2}(D^{k-1}f(z))\right]_{z=1} .$$

Since , for functions in $B_c(D)$, $D^k f(o)$ is real and non-negative, (5.1.2) may be written as

$$\sum_{n=2}^{\infty} n^2 \frac{D^{n+k}f(o)}{d_1 d_2 \cdots d_n} = \frac{D^{k+1}f(o)}{d_1} - \left[\frac{d}{dz}(D^k f(z))\right]_{z=1} - \left[\frac{d^2}{dz^2}(D^k f(z))\right]_{z=1}$$
$$\leq \frac{D^{k+1}f(o)}{d_1} .$$

Thus , in view of (5.2.9) ,

$$(5.2.10) \qquad V_{k+1}(f) \geqq o , \quad k = o, 1, 2, \ldots$$

if and only if , $f \in B_c(D)$.

Using the generating function $\Psi(zw)\left(\frac{2}{d_1} - (w\Psi'(w))'\right)^{-1}$, we define the sequence $\{q_n(z)\}_{n=o}^{\infty}$ by

$$(5.2.11) \qquad \sum_{n=o}^{\infty} q_n(z) w^n = \Psi(zw)\left(\frac{2}{d_1} - (w\Psi'(w))'\right)^{-1} .$$

The polynomials $q_n(z)$ are the generalization of Appell polynomials(c.f.Section 1.4) . To study the properties of the polynomial sequence $\{q_n(z)\}_{n=o}^{\infty}$, we define a sequence $\{v_n\}_{n=o}^{\infty}$ by

$$(5.2.12) \qquad \sum_{n=o}^{\infty} v_n z^n = \left(\frac{2}{d_1} - (w\Psi'(w))'\right)^{-1} .$$

This gives the recurrence relation

$$(5.2.13) \qquad v_o = d_1 \,,\; v_n = \sum_{k=o}^{n-1} (n-k+1)^2 \frac{v_k}{d_2 \cdots d_{n-k+1}}$$

so that $v_n > o$ for all n. Using (5.2.11),(5.2.12) and (5.2.13), we obtain

$$(5.2.14) \qquad q_n(o) = v_n \,,\; q_n(z) = \sum_{k=o}^{n} v_{n-k}\, e_k z^k .$$

Thus,

$$Dq_{n+1}(z) = q_n(z).$$

Let $Q_n(z)$ be the normalization of polynomials $q_n(z)$, i.e.,

$$(5.2.15) \qquad Q_n(z) = \frac{q_n(z) - q_n(o)}{e_1 \; Dq_n(o)}$$

$$= \frac{q_n(z)-q_n(o)}{e_1 v_{n-1}}$$

$$= \sum_{k=1}^{n} \frac{v_{n-k}}{v_{n-1}} \frac{e_k}{e_1} z^k .$$

We now find some preliminary properties of the sequence of linear functionals $\{U_k\}$ and $\{V_k\}$ in relation to the polynomial sequences $\{p_n(z)\}$ and $\{q_n(z)\}$. In fact, we show that the linear functionals $\{U_k\}$ and $\{V_k\}$ are biorthogonal to the polynomial sequences $\{p_n(z)\}$ and $\{q_n(z)\}$. Using this result, we find that the polynomial $P_n(z) \in B_s(D)$ $Q_n(z) \in B_c(D)$. We also show that the polynomials $\{P_n(z)\}$ and $\{Q_n(z)\}$ converge uniformly on compact subsets to functions in $B_s(D)$ and $B_c(D)$ respectively.

<u>Theorem 5.2.1</u> <u>The linear functionals</u>, $\{U_k\}_{k=1}^{\infty}$ <u>and</u> $\{V_k\}_{k=1}^{\infty}$ <u>are biorthogonal to the sequence of polynomials</u> $\{p_n(z)\}_{n=o}^{\infty}$ <u>and</u> $\{q_n(z)\}_{n=o}^{\infty}$.

<u>Proof</u>. For $n = 1, 2, \ldots$

$$(5.2.16) \qquad U_{k+1}(p_n) = \frac{2\, D^{k+1} p_n(o)}{d_1} - \left[\frac{d}{dz}(D^k p_n(z))\right]_{z=1}$$

$$= \frac{u_{n-k-1}}{d_1} - \left(2\,\frac{u_{n-k-2}}{d_1 d_2} + \ldots + (n-k)\frac{u_o}{d_1 d_2 \cdots d_{n-k}}\right)$$

$$= \begin{cases} o, & k \neq n-1 \\ 1, & k = n-1, \quad n = 1, 2, \ldots \end{cases}$$

This shows that $\{U_k\}_{k=1}^{\infty}$ is biorthogonal to $\{p_n(z)\}_{n=o}^{\infty}$.

Similarly, we have the biorthogonality relation concerning the sequence of linear functionals $\{V_k\}_{k=1}^{\infty}$ and the sequence of polynomials $\{q_n(z)\}_{n=o}^{\infty}$ as follows: For $n = 1, 2, \ldots$

$$(5.2.17) \quad V_{k+1}(q_n) = \frac{2D^{k+1} q_n(o)}{d_1} - \left[\frac{d}{dz}(D^k q_n(z))\right]_{z=1} - \left[\frac{d^2}{dz^2}(D^k q_n(z))\right]_{z=1}$$

$$= \frac{v_{n-k-1}}{d_1} - \left(2\,\frac{v_{n-k-2}}{d_1 d_2} + \ldots + (n-k)^2\,\frac{v_o}{d_1 d_2 \cdots d_{n-k}}\right)$$

$$= \begin{cases} o, & k \neq n-1 \\ 1, & k = n-1, \quad n = 1, 2, \ldots \end{cases}$$

This completes the proof.

Theorem 5.2.2 For each positive integer n, the polynomials $P_n(z) \in B_s(D)$, $2z-P_n(z) \in A_s(D)$. Similarly, $Q_n(z) \in B_c(D)$ and $2z-Q_n(z) \in A_c(D)$.

Proof. By (5.2.8),

$$U_{k+1}(P_n) = \frac{d_1}{p_{n-1}(o)} U_{k+1}(p_n).$$

Thus, from (5.2.16), we get for each positive integer n

$$U_{k+1}(P_n) = \begin{cases} o & , k \neq n-1 \\ \dfrac{d_1}{p_{n-1}(o)} & , k = n-1, \quad n = 1, 2, \ldots \end{cases}$$

which implies that $U_{k+1}(P_n) \geqq o$. Therefore, in view of (5.2.3), $P_n(z) \in B_s(D)$. Since $P_n(z)$, by (5.1.1), it is clear that $2z - P_n(z) \in A_s(D)$.

Also, by (5.2.15)

$$V_{k+1}(Q_n) = \frac{d_1}{q_{n-1}(o)} V_{k+1}(q_n).$$

Hence from (5.2.17), we deduce that

$$V_{k+1}(Q_n) = \begin{cases} o & , k \neq n-1 \\ \dfrac{d_1}{q_{n-1}(o)} & , k = n-1, \quad n = 1, 2, \ldots \end{cases}$$

Thus, $V_{k+1}(Q_n) \geqq o$ and in view of (5.2.10), the polynomial $Q_n(z) \in B_c(D)$. Further, by using (5.1.2), it is immediate that $2z-Q_n(z) \in A_c(D)$.

Define the function $F_s(z)$ and $F_c(z)$ by

$$(5.2.18) \qquad F_s(z) = \frac{\Psi(\hat{\beta}z)-1}{e_1 \hat{\beta}}, \quad F_c(z) = \frac{\Psi(\hat{\alpha}z)-1}{e_1\hat{\alpha}}$$

where $\hat{\beta}$ is the real root of (5.1.4) and $\hat{\alpha}$ is the real root of (5.1.5). Clearly, $F_s(z) \varepsilon B_s(D)$ and $F_c(z) \varepsilon B_c(D)$.

We now prove

Theorem 5.2.3 *We have*

$$\lim_{n\to\infty} P_n(z) = F_s(z) \text{ and } \lim_{n\to\infty} Q_n(z) = F_c(z)$$

uniformly on compact sets where, $F_s(z)$ *and* $F_c(z)$ *are defined by* (5.2.18).

Proof. We prove a general result. Let Φ be analytic at each point of a closed disc $|w| \leq R$, except for a simple pole at the point $w = w_o$, $o < |w_o| < R$. Let $\{a_n(z)\}_{n=o}^{\infty}$ be the polynomial sequence whose generating function is $\Phi(w)\Psi(zw)$, so that

$$(5.2.19) \qquad \Phi(w)\Psi(zw) = \sum_{n=o}^{\infty} a_n(z)w^n, \quad |w| < |w_o|.$$

For $n \geq 1$, let

$$(5.2.20) \qquad A_n(z) = \frac{a_n(z)-a_n(o)}{e_1 Da_n(o)} = \frac{a_n(z)-a_n(o)}{e_1 \; a_{n-1}(o)}.$$

From (5.2.19), we obtain the integral representation

$$a_n(z) = \frac{1}{2\pi i} \int_{|w|=\delta} \frac{\Phi(w)\Psi(zw)}{w^{n+1}} dw$$

which is valid if $o < \delta < |w_o|$. Now, on applying Residue Theorem to this, we get

(5.2.21) $$a_n(z) = \frac{w_1 \Psi(w_o z)}{w_o^{n+1}} + \frac{1}{2\pi i} \int_{|w|=R} \frac{\Phi(w)\Psi(zw)}{w^{n+1}}\, dw$$

where $w_1 = \lim_{w \to w_o} (w_o - w)\Phi(w)$. Let

$$\max_{|w| = R} |\Phi(w)| \leq M.$$

Then, from (5.2.21), we obtain

$$|a_n(z) w_o^{n+1} - w_1 \Psi(zw_o)| \leq MR\left(\frac{|w_o|}{R}\right)^{n+1} \Psi(R|z|).$$

This gives the asymptotic estimate

(5.2.22) $$a_n(z) w_o^{n+1} = w_1 \Psi(zw_o) + O(\varepsilon^n), \quad n \to \infty$$

where $\varepsilon = \frac{|w_o|}{R}$. The estimate (5.2.22) holds uniformly on every compact set. Setting $z = 0$ in (5.2.22), we have

(5.2.23) $$a_n(o) w_o^{n+1} = w_1 + O(\varepsilon^n), \quad n \to \infty$$

Using (5.2.20) and (5.2.23), we get

(5.2.24) $$A_n(z) = \frac{1}{e_1 w_o}(\Psi(w_o z) - 1) + O(\varepsilon^n), \quad n \to \infty.$$

The estimate (5.2.24) is uniform on every compact sets.

Now, we consider two special cases:

Let

(i) $\Phi(w) \equiv (\frac{2}{d_1} - \Psi'(w))^{-1}$. Then, in this case $w_o = \hat{\beta}$ and $A_n(z) = P_n(z)$, so that by (5.2.24)

$$\lim_{n \to \infty} P_n(z) = \frac{\Psi(\hat{\beta} z) - 1}{e_1 \hat{\beta}}$$

uniformly or compact sets.

(ii) Let $\Phi(w) \equiv \left(\frac{2}{d_1} - (w\Psi'(w))'\right)^{-1}$. Then, $w_o = \hat{\alpha}$ and $A_n(z) = Q_n(z)$. Therefore by (5.2.24), we have

$$\lim_{n\to\infty} Q_n(z) = \frac{\Psi(\hat{\alpha}z)-1}{e_1\hat{\alpha}}$$

uniformly on compact sets. This proves the theorem 5.2.3.

Remark : For $\Phi(w) \equiv \left(\frac{2}{d_1} - \Psi'(w)\right)^{-1}$, we have $w_o = \hat{\beta}$ and $a_n(o) = u_n$. Thus, from (5.2.23) we get,

$$\lim_{n\to\infty} u_n(\hat{\beta})^{n+1} = \frac{1}{\Psi''(\hat{\beta})} . \tag{5.2.25}$$

Similarly, for $\Phi(w) \equiv \left(\frac{2}{d_1} - (w\Psi'(w))'\right)^{-1}$, we deduce that $w_o = \hat{\alpha}$ and $a_n(o) = v_n$. Thus, by (5.2.23),

$$\lim_{n\to\infty} v_n(\hat{\alpha})^{n+1} = \frac{1}{2\Psi''(\hat{\alpha})+\hat{\alpha}\Psi'''(\hat{\alpha})} \tag{5.2.26}$$

5.3 The main purpose of this section is to find the sharp coefficient bounds for functions belonging to the class $A_s(D)$ and $A_c(D)$. As a consequence, it follows that functions in $A_s(D)$ and $A_c(D)$ are of finite Ψ-type. Two polynomial respresentations for entire functions of finite Ψ-type are developed in the process. We further show that certain entire functions can be decomposed as the sum of two G-L absolute starlike or two G-L absolute convex functions.

We need the following lemmas :

Lemma 5.3.1 <u>Every entire function f with Ψ-type less than $\hat{\beta}$ has a polynomial expansion of the form</u>

$$f(z) = \sum_{n=o}^{\infty} U_n(f) p_n(z) \tag{5.3.1}$$

<u>where $\hat{\beta}$ is the real root of the equation</u> (5.1.4) , U_n's <u>are defined by</u> (5.2.1) <u>and</u> $p_n(z)$'s <u>are given by</u> (5.2.7).

<u>Proof</u>. Since f is of Ψ-type less than $\hat{\beta}$, we have by (1.4.11)

$$f(z) = \sum_{n=o}^{\infty} \pounds_n(f) p_n(z) \tag{5.3.2}$$

where ,

$$\pounds_n(f) = \frac{1}{2\pi i} \int_{\Gamma} \frac{w^n}{2e_1 - \Psi'(w)} \hat{f}(w)\, dw$$

Γ is a circumference $|w| = \rho_o$ with $\rho_o < \hat{\beta}$ on which $(2e_1 - \Psi'(w))^{-1} \neq o$ and $\hat{f}(w)$ is the Laplace transform of $f(z)$.

Now , for k = o,1,2,...

$$2\,\frac{D^{k+1} f(o)}{d_1} = 2\,\frac{d_{k+1}\cdots d_1}{(k+1)!}\, D^{k+1} f(o) . \tag{5.3.3}$$

and

$$\left[\frac{d}{dz}(D^k f(z))\right]_{z=1} = (d_{k+1}\cdots d_2)\, a_{k+1} + 2\,\frac{d_{k+2}\cdots d_1}{d_2 d_1} a_{k+2} + \tag{5.3.4}$$

$$= \sum_{n=1}^{\infty} \frac{n}{d_n \cdots d_1} \frac{d_{n+k}\cdots d_1}{(n+k)!} f^{(n+k)}(o) .$$

But , from (1.4.5) , we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \Psi(zw) \hat{f}(w)\, dw .$$

This gives for $n = o, 1, 2, \ldots$

$$f^{(n+k)}(o) = \frac{(n+k)!}{2\pi i(d_1 \cdots d_{n+k})} \int_{\Gamma} w^{n+k} \hat{f}(w)\, dw$$

or,

$$\frac{d_1 \cdots d_{n+k}}{(n+k)!} f^{(n+k)}(o) = \frac{1}{2\pi i} \int_{\Gamma} w^{n+k} \hat{f}(w)\, dw .$$

Thus, (5.3.4) can be written as

$$(5.3.5) \quad \left[\frac{d}{dz}(D^k f(z)) \right]_{z=1} = \sum_{n=1}^{\infty} \frac{n}{d_1 d_2 \cdots d_n} \frac{1}{2\pi i} \int_{\Gamma} w^{n+k} \hat{f}(w)\, dw$$

$$= \frac{1}{2\pi i} \int_{\Gamma} w^{k+1} \left(\sum_{n=1}^{\infty} \frac{n}{d_1 \cdots d_n} w^{n-1} \right) \hat{f}(w)\, dw$$

$$= \frac{1}{2\pi i} \int_{\Gamma} w^{k+1} (2e_1 - \Psi'(w))^{-1} \hat{f}(w)\, dw.$$

Now, by using (5.3.3) and (5.3.5), we obtain for $k = o, 1, 2, \ldots$

$$(5.3.6) \quad U_{k+1}(f) = \frac{1}{2\pi i} \int_{\Gamma} \frac{w^{k+1}}{A(w)} \hat{f}(w)\, dw$$

where $A(w) = (2e_1 - \Psi'(w))$. Thus, for $k = o, 1, 2, \ldots,$

$$\pounds_{k+1}(f) = U_{k+1}(f)$$

and hence (5.3.2) gives

$$f(z) = \sum_{n=o}^{\infty} U_n(f) p_n(z).$$

This proves Lemma 5.3.1.

<u>Remarks 1</u>. Let f be an entire function with Ψ-type less than $\hat{\beta}$ and $f(o) = o$. Since, by (5.3.1),

$$o = f(o) = \sum_{n=o}^{\infty} U_n(f) p_n(z)$$

we have, by using (5.2.8),

$$f(z) = \sum_{n=0}^{\infty} U_n(f)(p_n(z)-p_n(0)) \tag{5.3.7}$$

$$= e_1 \sum_{n=1}^{\infty} U_n(f)u_{n-1}\ P_n(z)$$

$$= e_1 \sum_{n=0}^{\infty} U_{n+1}(f)u_n\ P_{n+1}(z)$$

2. The representation (5.3.1) is not valid for entire functions with Ψ-type $\hat{\beta}$. To see this, consider the function $F_s(z) = (\Psi(\hat{\beta}z)-1)/e_1\hat{\beta}$, which is entire with Ψ-type $\hat{\beta}$. Further, since $\Psi'(\hat{\beta}) = 2e_1$, for $n = 0,1,2,\ldots$

$$U_{n+1}(F_s) = 2\,\frac{D^{n+1}F_s(0)}{d_1} - \left[\frac{d}{dz}(D^nF_s(z))\right]_{z=1}$$

$$= 2\hat{\beta}^{\,n+1} - d_1\,\hat{\beta}^{\,n+1}\,\Psi'(\hat{\beta})$$

$$= 0.$$

Thus, the representation (5.3.1) is not valid for $F_s(z)$.

Next, we find a representation formula for entire functions with Ψ-type less than $\hat{\alpha}$ in terms of the polynomial sequence $\{q_n(z)\}_{n=0}^{\infty}$.

Lemma 5.3.2 *Every entire function f with Ψ-type less than $\hat{\alpha}$ has a polynomial expansion of the form*

$$f(z) = \sum_{n=0}^{\infty} V_n(f)q_n(z) \tag{5.3.8}$$

where $\hat{\alpha}$ is the root of the equation (5.1.5) *and* V_n, $q_n(z)$ *are defined by* (5.2.9) *and* (5.2.14) *respectively.*

Proof. Since the entire function f is of Ψ-type less than $\hat{\alpha}$, we have by (1.4.11)

$$(5.3.9) \qquad f(z) = \sum_{n=0}^{\infty} \mathcal{L}_n(f) q_n(z)$$

where

$$(5.3.10) \qquad \mathcal{L}_n(f) = \frac{1}{2\pi i} \int_{\Gamma} \frac{w^n}{2e_1 - (w\Psi'(w))'} \hat{f}(w)\, dw$$

and $\hat{f}$ is the Laplace transform of $f(z)$, Γ is a circumference $|w| = \rho_o$, $\rho_o < \hat{\alpha}$ on which $2e_1 - (w\Psi'(w))' \neq 0$.

For $k = 0, 1, 2, \ldots,$ we have

$$(5.3.11) \qquad \left[\frac{d^2}{dz^2}(D^k f(z))\right]_{z=1} = 2\, \frac{d_1 \cdots d_{k+2}}{d_1 d_2} a_{k+2} + 2\cdot 3 \frac{d_1 \cdots d_{k+3}}{d_1 d_2 d_3} a_{k+3} + \cdots$$

Now, following the same lines of proof that proved (5.3.6) and by using (5.3.3), (5.3.4) and (5.3.11), we obtain for $k = 0, 1, 2, \ldots$

$$\begin{aligned}
V_{k+1}(f) &= 2\, \frac{D^{k+1} f(o)}{d_1} - \left[\frac{d}{dz} D^k f(z)\right]_{z=1} - \left[\frac{d^2}{dz^2} D^k f(z)\right]_{z=1} \\
&= \sum_{n=1}^{\infty} n^2 \frac{d_1 \cdots d_{n+k}}{d_1 \cdots d_n} a_{n+k} \\
&= \sum_{n=1}^{\infty} n^2 \frac{d_1 \cdots d_{n+k}}{d_1 \cdots d_n} \frac{f^{(k+n)}(o)}{(n+k)!} \\
&= \sum_{n=1}^{\infty} \frac{n^2}{d_1 \cdots d_n} \frac{1}{2\pi i} \int_{\Gamma} w^{n+k} \hat{f}(w)\, dw \\
&= \frac{1}{2\pi i} \int_{\Gamma} w^{k+1} \left(\sum_{n=1}^{\infty} \frac{n^2}{d_1 \cdots d_n} w^{n-1} \right) \hat{f}(w)\, dw \\
&= \frac{1}{2\pi i} \int_{\Gamma} w^{k+1} \left(\frac{2}{d_1} - (w\Psi'(w))' \right) \hat{f}(w)\, dw
\end{aligned}$$

Thus , for $k = 0,1,2,\ldots$

$$\pounds_{k+1}(f) = V_{k+1}(f)$$

so that

$$f(z) = \sum_{n=0}^{\infty} V_n(f)q_n(z)\,.$$

This completes the proof of Lemma 5.3.2.

Remarks 1. Let f be an entire function with Ψ-type less than $\hat{\alpha}$ and $f(0) = 0$. Then , by (5.3.8),

$$0 = f(0) = \sum_{n=0}^{\infty} V_n(f)q_n(0)\,.$$

This gives by using (5.2.15),

$$f(z) = \sum_{n=0}^{\infty} V_n(f)(q_n(z)-q_n(0)) \tag{5.3.12}$$

$$= e_1 \sum_{n=1}^{\infty} V_n(f)v_{n-1}\, Q_n(z)$$

$$= e_1 \sum_{n=0}^{\infty} V_{n+1}(f)v_n\, Q_{n+1}(z)\,.$$

2. The representation (5.3.8) is not valid for every entire function with Ψ-type $\hat{\alpha}$. To illustrate this , consider the function $F_c(z)$, given by

$$F_c(z) = \frac{\Psi(\hat{\alpha}z)-1}{e_1\hat{\alpha}}$$

where $\hat{\alpha}$ is the real root of the equation given by (5.1.5). Clearly , $F_c(z)$ is an entire function with Ψ-type $\hat{\alpha}$. However ,

$$V_{n+1}(F_c) = 2\,\frac{D^{n+1}F_c(0)}{d_1} - \left[\frac{d}{dz}(D^nF_c(z))\right]_{z=1} - \left[\frac{d^2}{dz^2}(D^nF_c(z))\right]_{z=1}$$

$$= 2\,\hat{\alpha}^{n+1} - d_1\,\hat{\alpha}^{n+1}\,\Psi(\hat{\alpha}) - d_1\,\hat{\alpha}^{n+1}$$

$$= 0\,.$$

Therefore , the representation (5.3.8) is not valid for $F_c(z)$.

Next , we find the sharp coefficient bounds for functions belonging to the classes $A_s(D)$ and $A_c(D)$ respectively.

<u>Theorem 5.3.1 Let</u> m <u>be a positive integer greater than</u> 1. <u>If</u> $f(z) = z + \sum_{m=2} a_m z^m$ <u>belongs to</u> $A_s(D)$, <u>then</u>

$$(5.3.13) \qquad |a_m| \leq \frac{u_o}{(d_2 \cdots d_m) u_{m-1}} .$$

<u>Equality holds , if and only if ,</u> $f \equiv P_n$ <u>given by</u> (5.2.8).
<u>Further , if</u> $f \in A_c(D)$, <u>then</u>

$$(5.3.14) \qquad |a_m| \leq \frac{v_o}{(d_2 \cdots d_m) v_{m-1}} .$$

<u>Equality holds if and only if</u> $f \equiv Q_n$, <u>given by</u> (5.2.15).

<u>Proof</u>. Let m be a positive integer greater than 1 and suppose that $f \in A_s(D)$. Clearly , by (5.1.1), the partial sum

$$f_m(z) = z + \sum_{n=2}^{m} \frac{D^n f(o)}{d_1 \cdots d_n} z^n$$

belongs to $A_s(D)$. Therefore ,

$$g_m(z) = z + \sum_{n=2}^{m} \frac{|D^n f(o)|}{d_1 \cdots d_n} z^n$$

belongs to $A_s(D)$ and satisfies $D^m g_m(o) = |D^m f(o)|$. Further , $g_m(z)$ is an entire function with Ψ-type zero, i.e., less than β . Since , $U_n(g_m) = o$ for $n \geq m+1$, using the expansion formula (5.3.7) for $f \equiv g_m$, we obtain

(5.3.15) $$q_m(z) = e_1 \sum_{n=o}^{\infty} U_{n+1}(q_m)u_n \, P_{n+1}(z)$$
$$= e_1 \sum_{n=o}^{m-1} U_{n+1}(q_m)\, u_n \, P_{n+1}(z).$$

Since $Dq_m(o) = d_1$ and $DP_{n+1}(o) = d_1$, taking the Gelfond-Leontev derivative in (5.3.15) and setting $z = o$ in the resulting equation, we get

$$d_1 = \sum_{n=o}^{m-1} U_{n+1}(q_m)u_n.$$

The numbers $U_{n+1}(q_m)$, $o \leqq n \leqq m-1$, are non-negative ;therefore

$$U_m(q_m)u_{m-1} \leqq d_1 .$$

Again, taking the Gelfond-Leontev derivative in (5.3.15) m times and setting $z = o$, we have

$$|a_m| = \frac{D^m q_m(o)}{d_1 \cdots d_m} = U_m(q_m)u_{m-1}e_1 \frac{D^m P_m(o)}{d_1 \cdots d_m}$$
$$\leqq \frac{D^m P_m(o)}{d_1 \cdots d_m}$$
$$\leqq \frac{u_o}{(d_2 \cdots d_m)u_{m-1}} .$$

This gives the estimate (5.3.13). The proof of (5.3.14) follows by using the representation (5.3.12) and by following the same lines of proof that proved (5.3.13).

By Theorem 5.2.2, the polynomial $P_n(z) \varepsilon B_s(D)$ and $Q_n(z) \varepsilon B_c(D)$. It is easily seen that equality holds in (5.3.13) and (5.3.14), respectively, for the polynomials $P_n(z)$ and $Q_n(z)$.

<u>Corollary</u> 5.3.1 <u>If</u> $f \in A_s(D)$, <u>then</u> f <u>is of finite</u> Ψ-<u>type atmost</u> $\hat{\beta}$ <u>where</u> , $\hat{\beta}$ <u>is the real root of the equation given by</u> (5.1.4).

<u>Proof</u>. Since $f \in A_s(D)$, we have from (5.3.13) for $n = 1, 2, \ldots$

$$D^n f(o) \leq \frac{u_o d_1}{u_{n-1}} .$$

Consequently , by (5.2.25) ,

$$\limsup_{n\to\infty} |D^n f(o)|^{1/n} \leq \hat{\beta}$$

Thus , in view of (1.3.16) , f is of Ψ-type atmost $\hat{\beta}$.

<u>Corollary</u> 5.3.2 <u>If</u> $f \in A_c(D)$, <u>then</u> f <u>is of</u> Ψ-<u>type atmost</u> $\hat{\alpha}$ <u>where</u> , $\hat{\alpha}$ <u>is the real of the equation given by</u> (5.1.5).

<u>Proof</u>. From (5.3.14) , we get

$$D^n f(o) \leq \frac{v_o d_1}{v_{n-1}} .$$

Then , by using (5.2.26) , we deduce that

$$\limsup_{n\to\infty} |D^n f(o)|^{1/n} \leq \hat{\alpha}$$

Thus , in view of (1.3.16) , f is of Ψ-type atmost $\hat{\alpha}$.

<u>Remarks</u> 1. All entire function with finite Ψ-type need not be G-L absolute starlike , the function

$$F(z) = z + \sum_{n=2}^{\infty} (-1)^n \frac{1}{d_1 d_2 \cdots d_{2n+1}} z^{2n+1}$$

where $d_n \to \infty$ as $n \to \infty$, is entire with Ψ-type 1 but is not G-L absolute starlike.

Theorem 5.3.3 *Every entire function f satisfying (5.3.16) can be expressed as the sum of two G-L absolute starlike functions. Every entire function f satisfying (5.3.17) can be expressed as the sum of two G-L absolute convex functions.*

To prove this theorem, we need the following result which we state in the form of a lemma.

Lemma 5.3.3 [17]. *Let M denote the family of power series* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *satisfying* $|a_n| \leq m_n$ *where,* $\{m_n\}_{n=0}^{\infty}$ *is a sequence of positive numbers. Let* M_o *be the subclass of M for which* $|a_n| = M_n$, $n = 0, 1, 2, \ldots$. *Then, for every* $f \in M$

$$f(z) = \tfrac{1}{2}g(z) + \tfrac{1}{2}h(z)$$

where g and h are functions in M_o.

Proof of Theorem 5.3.3 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be any entire function satisfying (5.3.16). This implies that there exists a positive constant k_1 such that

(5.3.18) $$|D^n f(o)| \leq 2k_1 \hat{\beta}^n, \quad n = 0, 1, 2, \ldots .$$

Let $m_o = k_1$ and $m_n = k_1 \hat{\beta}^n / (d_1 \cdots d_n)$, $n = 1, 2, \ldots$. Then, by Lemma 5.3.3,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{2} \sum_{n=0}^{\infty} g_n z^n + \frac{1}{2} \sum_{n=0}^{\infty} h_n z^n$$
$$= \tfrac{1}{2}g(z) + \tfrac{1}{2}h(z)$$

where $|h_o| = |g_o| = k_1$, $|g_n| = |h_n| = k_1 \hat{\beta}^n / (d_1 \cdots d_n)$, $n = 1, 2, \ldots$

Now, in view of (5.1.1) and the fact that $\Psi'(\hat{\beta}) = 2e_1$, the function g(z) and h(z) are G-L absolute starlike. This proves the first half of the theorem.

On replacing $\hat{\beta}$ by $\hat{\alpha}$ in (5.3.18), applying Lemma 5.3.3 and following the same lines of proof, the second part of the theorem follows. This completes the proof of Theorem 5.3.3.

Remarks 1. Let f be an entire function. If

$$\lim_{j\to\infty} \frac{D^j f(o)}{\hat{\beta}^j} = o \tag{5.3.19}$$

then for infinitely many k, we have

$$\frac{D^{k+1} f(o)}{\hat{\beta}^{k+1}} \geq \sup_{n\geq 2} \frac{D^{n+k} f(o)}{\hat{\beta}^{n+k}} .$$

Since $\Psi'(\beta) = 2e_1$, for these integers k, we deduce that

$$\sum_{n=2}^{\infty} n \frac{|D^{n+k} f(o)|}{d_1 d_2 \cdots d_n} \leq \frac{|D^{k+1} f(o)|}{d_1} .$$

This implies that $D^k f(\Delta)$ is starlike with respect to the point $D^k f(o)$ for infinitely many k. Observe that (5.3.19) holds for every entire function with Ψ-type less than β.

Similar is the case for convexity. If

$$\lim_{j\to\infty} \frac{D^j f(o)}{\hat{\alpha}^j} = o \tag{5.3.20}$$

then $D^k f(\Delta)$ is convex for infinitely many k. We note that (5.3.20) holds for every entire function with Ψ-type less than $\hat{\alpha}$.

2. The subclasses $B_s(D)$ and $B_c(D)$ are clearly convex. However, this is not true for the larger classes $A_s(D)$ and $A_c(D)$. To illustrate this, let the functions F and G be defined by

$$F(z) = z + \frac{2d_3}{(4d_3+3d_2)} z^2 + \frac{d_2}{(4d_3+3d_2)} z^3$$

and

$$G(z) = z - \frac{2d_3}{(4d_3+3d_2)} z^2 + \frac{d_2}{(4d_3+3d_2)} z^3.$$

Clearly, by (5.1.1), F and G belongs to $A_s(D)$. But, $tF+(1-t)G$ fails to belong to $A_s(D)$ for every t in $(o,1)$.

Similarly, if we define the functions F_o and G_o by

$$F_o(z) = z + \frac{4d_3}{(16d_3+9d_2)} z^2 + \frac{d_2}{(16d_3+9d_2)} z^3$$

$$G_o(z) = z - \frac{4d_3}{(16d_3+9d_2)} z^2 + \frac{d_2}{(16d_3+9d_2)} z^3$$

then F_o and G_o belong to $A_c(D)$ but $tF_o+(1-t)G_o \notin A_c(D)$ for every t in $(o,1)$.

5.4 In this section, we find the simplicial representation formulae for functions belonging to the classes $B_s(D)$ and $B_c(D)$.

We need the following lemma.

Lemma 5.4.1 If f belongs to $B_s(D)$, then

$$(5.4.1) \qquad \sum_{k=o}^{\infty} U_{k+1}(f)u_k \leq d_1 .$$

If f belongs to $B_c(D)$, then

$$(5.4.2) \qquad \sum_{k=o}^{\infty} V_{k+1}(f)v_k \leq d_1 .$$

Proof. Suppose $f \in B_s(d)$. For each $k \geq o$, we have

$$(5.4.3) \qquad U_{k+1}(f) = 2\,\frac{D^{k+1}f(o)}{d_1} - \left[\frac{d}{dz}(D^k f(z))\right]_{z=1}$$

$$= \frac{D^{k+1}f(o)}{d_1} - \sum_{j=k+1}^{\infty} \frac{(j-k+1)}{d_1 \cdots d_{j-k+1}} D^{(j+1)}f(o).$$

Then, by using (5.4.3),

$$\sum_{k=o}^{n} U_{k+1}(f)u_k = \sum_{k=o}^{n} \frac{D^{k+1}f(o)}{d_1}u_k - \sum_{k=o}^{n} u_k \sum_{j=k+1}^{\infty} \frac{(j-k+1)}{d_1 \cdots d_{j-k+1}} D^{j+1}f(o).$$

Now,

$$\sum_{k=o}^{n} u_k \sum_{j=k+1}^{\infty} \frac{(j-k+1)}{d_1 \cdots d_{j-k+1}} D^{j+1}f(o) = \sum_{j=1}^{\infty} D^{j+1}f(o) \sum_{k=o}^{\min\{n,k-1\}} \frac{(j-k+1)}{d_1 \cdots d_{j-k+1}} u_k .$$

Spliting the sum on the right hand side of the above equation at $j = n$, we obtain, in view of the recurrence formula (5.2.6),

$$\sum_{j=1}^{n} D^{j+1}f(o) \sum_{k=o}^{j-1} \frac{(j-k+1)}{d_1 \cdots d_{h-k+1}} u_k + \sum_{j=n+1}^{\infty} D^{j+1}f(o) \sum_{k=o}^{n} \frac{(j-k+1)}{d_1 \cdots d_{j-k+1}} u_k$$

$$= \sum_{j=1}^{n} \frac{D^{j+1}f(o)}{d_1} u_j + \sum_{j=n+1}^{\infty} D^{j+1}f(o) \sum_{k=o}^{n} \frac{(j-k+1)}{d_1 \cdots d_{j-k+1}} u_k .$$

Thus,

$$\sum_{k=o}^{n} U_{k+1}(f)u_k = \frac{Df(o)}{d_1} - \sum_{j=n+1}^{\infty} D^{j+1}f(o) \sum_{k=o}^{n} \frac{(j-k+1)}{d_1 \cdots d_{j-k+1}} u_k .$$

$$\leq d_1 .$$

This proves (5.4.1). The inequality in (5.4.2) can be proved by using (5.2.9) and following the same lines of proof that proved (5.4.1).

We now prove.

<u>Theorem</u> 5.4.1 <u>If</u> $f \in B_s(D)$, <u>then</u>

$$f(z) = e_1 S(f) F_s(z) + e_1 \sum_{k=0}^{\infty} U_{k+1}(f) u_k P_{k+1}(z) \tag{5.4.4}$$

<u>where</u>,

$$S(f) = d_1 - \sum_{k=0}^{\infty} U_{k+1}(f) u_k$$

<u>The expansion is valid for all</u> z <u>and the convergence is uniform on every compact set</u>.

<u>Proof</u>. Set,

$$F(z) = e_1 \sum_{k=0}^{\infty} U_{k+1}(f)\, u_k\, P_{k+1}(z). \tag{5.4.5}$$

Then in view of Theorem 5.2.3 and (5.4.1), the series in (5.4.5) converges uniformly on compact sets. Further, we have

$$|F(z)| \leq \sum_{k=0}^{\infty} U_{k+1}(f)\, u_k\, |P_{k+1}(z)|$$

$$\leq K_1 \, F_s(|z|)$$

where K_1 is a constant. This gives that $F(z)$ is of Ψ-type atmost $\hat{\beta}$. Since f belongs to $B_s(D)$, by Corollary 5.3.1, f is entire function of Ψ-type not exceeding $\hat{\beta}$. Therefore, the function

$$g(z) = f(z) - F(z) \tag{5.4.6}$$

is an entire function of Ψ-type atmost $\hat{\beta}$. Since, by (5.2.16)

$$U_{n+1}(P_k) = \begin{cases} o & , k \neq (n+1) \\ \dfrac{d_1}{P_{k-1}(o)} & , k = n+1 , n = 1, 2, \ldots \end{cases}$$

on applying the linear functional U_{n+1} in (5.4.6), we obtain

$$U_{n+1}(g) = o , \quad n = o, 1, 2, \ldots$$

The function $2e_1 - \Psi'(z)$ is regular in $|z| \leq \hat{\beta}$ and has only one simple zero in $|z| \leq \hat{\beta}$. Further, since g is an entire function of Ψ-type atmost $\hat{\beta}$, we have

$$\limsup_{n \to \infty} |D^n g(o)|^{\frac{1}{n}} \leq \hat{\beta}.$$

Thus, in view of Perron's theorem [61] we deduce that $Dg(z)$ is a constant multiple of $\Psi(\hat{\beta}z)$. That is,

$$Dg(z) = c_1 \Psi(\hat{\beta}z)$$

where $\hat{\beta}$ is the real root of the equation given by (5.1.4).

Since,

$$\begin{aligned} Dg(o) &= Df(o) - DF(o) \\ &= d_1 - \sum_{k=o}^{\infty} U_{k+1}(f) u_k \\ &= S(f) \ (\text{say}) \end{aligned}$$

and $g(o) = o$, we must have

(5.4.7) $$\begin{aligned} g(z) &= e_1 S(f) \frac{\Psi(\hat{\beta}z) - 1}{e_1 \hat{\beta}} \\ &= e_1 S(f) F_s(z). \end{aligned}$$

Now, combining (5.4.6) and (5.4.7), we get (5.4.4). This proves the theorem.

<u>Corollary</u> 5.4.1 <u>If</u> $f \in A_s(D)$, <u>then</u> <u>for</u> $|z| = r$

$$2r - \sup\{F_s(r), P_n(r); n=1,2,\ldots\} \leq |f(z)| \leq \sup\{F_s(r), P_n(r); n=1,2,\ldots\}$$

<u>Proof</u>. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $A_s(D)$. Then for $|z| = r$

$$|f(z)| \leq r + \sum_{n=2}^{\infty} |a_n| r^n$$

$$= r + \sum_{n=2}^{\infty} \frac{|D^n f(o)|}{d_1 \cdots d_n} r^n .$$

Since $f \in A_s(D)$, the function

$$g(z) = z + \sum_{n=2}^{\infty} \frac{|D^n f(o)|}{d_1 \cdots d_n} z^n$$

belongs to $B_s(D)$. Therefore, from (5.4.4), we have for $|z| = r$

$$|g(z)| \leq \sup\{F_s(r), P_n(r), n = 1,2,\ldots\} .$$

Thus, by using this, we get for functions in $A_s(D)$,

(5.4.8) $$|f(z)| \leq \sup\{F_s(r), P_n(r), n = 1,2,\ldots\} .$$

To obtain the lower bound of $|f(z)|$ for f in $A_s(D)$, we observe that the function $F(z) = 2z - f(z)$ belongs to $A_s(D)$. Therefore,

$$|f(z)| \geq 2|z| - |F(z)| .$$

Applying (5.4.8) to the function F, the above inequality yields for $|z| = r$

(5.4.9) $$|f(z)| \geq 2r - \sup\{F_s(r), P_n(r), n = 1,2,\ldots\} .$$

This completes the proof of Corollary 5.4.1.

Remark. Theorem 5.4.1 shows that the class $B_s(D)$ is the infinite dimensional simplex whose vertices are $F_s(z)$ and the polynomials $\{P_n(z)\}_{n=0}^{\infty}$.

The next theorem gives the simplicial representation formula for functions in the class $B_c(D)$.

Theorem 5.4.2 *If f belongs to $B_c(D)$, then*

$$(5.4.10) \quad f(z) = e_1 C(f)\, F_c(z) + e_1 \sum_{n=0}^{\infty} V_{n+1}(f) v_n\, Q_{n+1}(z)$$

where

$$C(f) = e_1 - \sum_{n=0}^{\infty} V_{n+1}(f) v_n.$$

The expansion is valid for all z and the convergence is uniform on every compact set.

Proof. The proof follows by using (5.4.2) and the fact that the function $2e_1-(w\Psi'(w))'$ has only one zero in the disc $|z| \leq \hat{\alpha}$, where $\hat{\alpha}$ is the real root of the equation given by (5.1.5).

Corollary 5.4.2. *If f belongs to $A_c(D)$, then for $|z| = r$*

$$2r-\sup\{F_c(r), Q_n(r), n = 1,2,\} \leq |f(z)| \leq \sup\{F_c(r), Q_n(r); m=1,2,\ldots\}$$

Proof. Since, from (5.4.10), for functions f in $B_c(D)$, we have for $|z| = r$

$$|f(z)| \leq \sup\{F_c(r), Q_n(r) ; n = 1,2,\ldots\}$$

the proof follows by using the same lines of proof as in Corollary 5.4.1.

Remark. Theorem 5.4.2 gives that the class $B_c(D)$ is the infinite dimensional simplex whose vertices are $F_c(z)$ and the polynomials $\{Q_n(z)\}_{n=0}^{\infty}$.

CHAPTER VI

COEFFICIENT ESTIMATES FOR GENERALIZED SPIRALLIKE FUNCTIONS AND RELATED FUNCTION CLASSES

6.1 The definitions of starlike functions with respect to symmetric points and spirallike functions (c.f. Section 1.2) motivate us to introduce analogous classes of analytic functions related to Gelfond-Leontev derivatives. In this chapter, we are primarily concerned with the study of coefficient bounds of functions in such classes.

Definition 6.1.1 A function $f \in H$ is said to be Generalized λ-spirallike function of order α with respect to N-symmetric points ($|\lambda| < \pi/2$, $0 \leq \alpha < 1$) if $Df(z)$ is analytic in Δ and for $z \in \Delta$,

$$(6.1.1) \qquad \mathrm{Re}\left\{ e^{i\lambda} z \frac{Df(z)}{d_1 f_N(z)} \right\} > \alpha \cos\lambda$$

where $f_N(z)$ is given by (1.2.20).

We denote the class of Generalized λ-spirallike functions of order α with respect to N-symmetric points by $S_D(\lambda, N, \alpha)$.

For $d_n \equiv n$, $S_D(\lambda, 1, 0)$ is the class of λ-spirallike functions introduced and studied by Spacek [100]. Similarly, for $d_n \equiv n$, $S_D(\lambda, 1, \alpha)$ is the class of λ-spirallike functions of order α studied by Libera [50] and $S_D(0, 1, \alpha)$ is the class of starlike functions of order α (c.f.(1.2.14)). For $d_n \equiv n$, the classes $S_D(0, N, \alpha)$ and $S_D(\lambda, N, \alpha)$ are recently studied in [20],[99] and [67].

Replacing z by $\mu_o z, \mu_o^2 z, \ldots, \mu_o^{N-1} z$ in (6.1.1) and adding the N equations, we have

$$\mathrm{Re}\ \{ e^{i\lambda}\ z\ \frac{Df_N(z)}{d_1 f_N(z)} \} > \alpha\cos\lambda$$

which implies that $f_N(z) \in S_D(\lambda, N, \alpha)$.

<u>Definition</u> 6.1.2 <u>A function</u> $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in H$ is said to be <u>Generalized</u> λ-<u>spirallike</u> <u>function</u> <u>of</u> <u>order</u> α if $Df(z)$ is analytic in Δ and for $z \in \Delta$ and $|\lambda| < \pi/2$

$$(6.1.2) \qquad \mathrm{Re}\ \{ e^{i\lambda}\ z\ \frac{Df(z)}{d_1 f(z)} \} > \alpha\cos\lambda .$$

We denote the class of Generalized λ-spirallike functions $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ of order α by $S_D^k(\lambda, \alpha)$ and observe that $S_D^1(\lambda, \alpha) \equiv S_D(\lambda, 1, \alpha)$.

Generalized λ-spirallike functionsof order $\alpha = o$ are called Generalized λ-spirallike functions. With $d_n \equiv n$ and for different values of the parameters λ, k and α, this class includes several known subclasses of univalent functions studied in [50], [100], [101].

Now, we define a subclass of $S_D^k(\lambda, \alpha)$ consisting of functions f for which $zDf(z)/d_1 f(z)$ is bounded.

<u>Definition</u> 6.1.3 A function $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in H$ is said to be <u>Generalized</u> λ-<u>spirallike</u> function of order α and type β, if $Df(z)$ is analytic in Δ and satisfies the condition

$$(6.1.3) \quad \left| \frac{\frac{zDf(z)}{d_1 f(z)} - 1}{2\beta\{\frac{zDf(z)}{d_1 f(z)} - 1 + (1-\alpha)\cos\lambda e^{-i\lambda}\} - (\frac{zDf(z)}{d_1 f(z)} - 1)} \right| < 1$$

for $|\lambda| < \pi/2$, $0 \leqq \alpha < 1$, $0 < \beta \leqq 1$ and $z \in \Delta$. Denote by, $S_D^k(\lambda, \alpha, \beta)$ the class of Generalized λ-spirallike function of order α and type β.

We observe that if f satisfies (6.1.3), then the image of Δ under $zDf(z)/d_1 f(z)$ lies in the disc with centre

$$1-(2\beta-1)((2\beta-1)-2\beta(1-\alpha)\cos^2\lambda)-i\beta(2\beta-1)(1-\alpha)\sin 2\lambda/(1-(2\beta-1)^2)$$

and radius

$$2\beta(1-\alpha)\cos\lambda/(1-(2\beta-1)^2).$$

For $d_n \equiv n$ and different values of the parameters α, β, k and λ, the class $S_D^k(\lambda, \alpha, \beta)$ includes several recently well studied subclasses of univalent functions ([27],[40],[48],[50],[53],[58]).

Section 6.2 deals with the determination of the coefficient estimates for Generalized λ-spirallike functions of order α with respect to N-symmetric points. We also find the influence of the fixed (N+1)th coefficient on the growth of other coefficients for functions belonging to this class. The coefficient estimates for functions $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ in the class $S_D^k(\lambda, \alpha)$ are found in Section 6.3. This section also contains the study of the effect of the coefficients $a_{k+1}, \ldots, a_{2k}$ on the bound of other coefficients of functions in $S_D^k(\lambda, \alpha)$. For $k = 1$, this reduces to the investigation of influence of fixed second coefficient on the remaining coefficients of f. Finally in Section 6.4,

the coefficient bounds for Generalized λ-spirallike functions with gaps k and having order α and type β are found.

6.2 This section deals with the determination of coefficient bounds for functions belonging to the class of Generalized λ-spirallike functions with respect to N-symmetric points. We also study the influence of the fixed (N+1)th coefficient on the growth of the other coefficient of Generalized λ-spirallike functions with respect to N-symmetric points.

We prove

<u>Theorem</u> 6.2.1 <u>Let</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_D(\lambda, N, \alpha)$. <u>Then</u>,

$$(6.2.1) \qquad |a_{mN+1}| \leq \prod_{j=o}^{m-1} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{jN+1}-d_1)|}{d_{((j+1)N+1)}-d_1}$$

<u>and</u>

$$(6.2.2) \qquad |a_{mN+p}| \leq \frac{2d_1(1-\alpha)\cos\lambda}{d_{mN+p}} \prod_{j=1}^{m} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{jN+1}-d_1)|}{d_{(j\quad N+1)}-d_1}$$

<u>for</u> $m = 1, 2, \ldots,\ p = 2, 3, \ldots, N$.

<u>Proof</u>. Let,

$$(6.2.3) \qquad w(z) = \frac{e^{i\lambda}\left(\frac{zDf(z)}{d_1 f_N(z)} - 1\right)}{e^{i\lambda}\frac{zDf(z)}{d_1 f_N(z)} + (e^{-i\lambda} - 2\alpha\cos\lambda)} = \sum_{n=1}^{\infty} w_n z^n .$$

Since, $f \in S_D(\lambda, N, \alpha)$, we have $\mathrm{Re}\left(e^{i\lambda} z\frac{Df(z)}{d_1 f_N(z)}\right) > \alpha\cos\lambda$. Therefore,

$|w(z)| < 1$ in Δ. We rewrite $f_N(z)$ given by (1.2.20) as

$$f_N(z) = z + \sum_{n=2}^{\infty} \delta_n z^n$$

where,

(6.2.4) $$\delta_n = \begin{cases} 1 , & n = NP+1 , \quad P = 1, 2, \ldots \\ o , & \text{otherwise}. \end{cases}$$

Clearly, $\delta_n^2 = \delta_n$.

Now, (6.2.3) gives

$$zDf(z) - d_1 f_N(z) = [zDf(z) + d_1 f_N(z)(e^{-2i\lambda} - 2\alpha\cos\lambda e^{-i\lambda})]w(z).$$

Replacing $f(z)$, $f_N(z)$ and $w(z)$ by their power series representation in the above equation, we obtain

(6.2.5) $$\sum_{n=2}^{\infty} (d_n - \delta_n d_1) a_n z^n$$
$$= [\sum_{n=1}^{\infty} \{d_n + d_1 e^{-i\lambda}(e^{-i\lambda} - 2\alpha\cos\lambda)\delta_n\} a_n z^n](\sum_{n=1}^{\infty} w_n z^n)$$

where, $a_1 = 1$.

Comparing the coefficient of z^n on both side of (6.2.5), we have

(6.2.6) $$(d_n - \delta_n d_1) a_n = \sum_{k=1}^{n-1} \{ d_k + d_1 e^{-i\lambda}(e^{-i\lambda} - 2\alpha\cos\lambda)\delta_k\} a_k w_{n-k}.$$

Thus, the coefficient a_n on left hand side of (6.2.6) depends upon the coefficients $a_1, a_2, \ldots, a_{n-1}$. Therefore, for suitable c_k's (6.2.5) may be written as

$$\sum_{k=2}^{n} (d_k - \delta_k d_1) a_k z^k + \sum_{k=n+1}^{\infty} c_k z^k$$
$$= [\sum_{k=1}^{n-1} \{ d_k + d_1 e^{-i\lambda}(e^{-i\lambda} - 2\alpha\cos\lambda)\delta_k \} a_k z^k] w(z)$$

where, $z = re^{i\theta}$.

Multiplying each side of this identity by its conjugate , integrating with respect to θ from o to 2π and taking limit as $r \to 1$, we get ,

$$\sum_{k=2}^{n} (d_k-\delta_k d_1)^2 |a_k|^2 \leq d_1^2|1+e^{-2i\lambda}-2\alpha\cos\lambda e^{-i\lambda}|^2+ \\ + \sum_{k=2}^{n-1} |d_k+d_1 e^{-i\lambda}(e^{-i\lambda}-2\alpha\cos\lambda)|^2|a_k|^2$$

and so

$$(d_n-\delta_n d_1)^2|a_n|^2 \leq d_1^2|1+e^{-2i\lambda}-2\alpha\cos\lambda e^{-i\lambda}|^2 \\ + \sum_{k=2}^{n-1} \{|d_k+d_1 e^{-i\lambda}(e^{-i\lambda}-2\alpha\cos\lambda)\delta_k|^2-(d_k-\delta_k d_1)^2\}|a_k|^2$$

Since $\delta_k^2 = \delta_k$, the above inequality after some simplifications reduces to

$$(6.2.7)\quad (d_n-\delta_n d_1)^2|a_n|^2 \leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+\sum_{k=2}^{n-1}(d_k-c.d_1)\delta_k|a_k|^2].$$

Now, we first prove (6.2.1) by using induction on m.

When $n = N+1$, (6.2.7) gives

$$(6.2.8)\qquad |a_{N+1}| \leq \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}|}{d_{N+1}-d_1} .$$

This shows that (6.2.1) is true for $m = 1$.

Next , assume that (6.2.1) is true for $m = 1, 2, \ldots, (q-1)$. For $n = qN+1$, since the contribution on the right hand side of (6.2.7) comes only from the terms corresponding to $k = N+1, 2N+1, \ldots, (q-1)N+1$, therefore by using the induction hypothesis , we have

$$(d_{qN+1}-d_1)^2|a_{qN+1}|^2$$

$$\leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+\sum_{j=1}^{q-1}(d_{jN+1}-\alpha d_1)\prod_{k=o}^{j-1}\frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{kN+1}-d_1)|}{(d_{((k+1)N+1)}-d_1)}$$

$$=[(d_{qN+1}-d_1)\prod_{j=o}^{q-1}\frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{jN+1}-d_1)|}{d_{((j+1)N+1)}-d_1}]^2.$$

The equality can be proved by induction on q. This proves (6.2.1) for m = q.

The inequality (6.2.2) is also proved by using induction on m.

When n = N+p , p = 2, 3, ..., N , (6.2.7) and (6.2.8) after a simple calculation gives

$$d_{N+p}^2|a_{N+p}|^2 \leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+(d_{N+1}-\alpha d_1)|a_{N+1}|^2]$$

$$\leq |2d_1(1-\alpha)\cos\lambda e^{-i\lambda}|^2 \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{N+1}-d_1)|^2}{(d_{N+1}-d_1)^2}.$$

Consequently,

$$|a_{N+p}|\leq\frac{(d_{N+1}-d_1)}{d_{N+p}}\prod_{j=o}^{1}\frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{jN+1}-d_1)|}{(d_{N+1}-d_1)}$$

so that (6.2.2) holds for m = 1 . Now, let (6.2.2) is true for m = 1, ..., q-1 . Thus, for n = qN+p the contribution on the right hand side of (6.2.7) comes only from the terms corresponding

to $k = N+1, 2N+1, \ldots, (q-1)N+1$, therefore by using the induction hypothesis in (6.2.7) as in the proof of (6.2.1) we get

$$|a_{qN+p}| \leq \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}|}{d_{qN+p}} \prod_{j=1}^{q} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{jN+1}-d_1)|}{(d_{jN+1}-d_1)}.$$

This proves (6.2.2) for $m = q$ and the proof of the theorem is complete.

<u>Remark</u>. For $d_n = n$, the estimate in (6.2.1) is sharp , the equality being attained for the function

$$F(z) = \frac{1}{(1-z^N)^{2(1-\alpha)\cos\lambda\ \exp(-i\lambda)/N}}$$

in the class $S_D(\lambda, N, \alpha)$.

Putting $\lambda = o$ in (6.2.1) and (6.2.2) we have the following.

<u>Corollary</u> 6.2.1. <u>Let</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_D(o, N, \alpha)$. <u>Then</u>

$$(6.2.9) \qquad |a_{mN+1}| \leq \prod_{j=o}^{m-1} \frac{2d_1(1-\alpha)+(d_{jN+1}-d_1)}{d_{((j+1)N+1)}-d_1}$$

<u>and</u>

$$(6.2.10) \quad |a_{mN+p}| \leq \frac{2d_1(1-\alpha)}{d_{mN+p}} \prod_{j=1}^{m} \frac{2d_1(1-\alpha)+(d_{jN+1}-d_1)}{(d_{jN+1}-d_1)}$$

<u>for</u> $m = 1, 2, \ldots,$ and $p = 2, 3, \ldots, N$.

Our next theorem gives the influence of fixed (N+1)th coefficient on the growth of other coefficients of Generalized λ-spirallike functions with respect to N-symmetric points.

Theorem 6.2.2 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_D(\lambda, N, \alpha)$. Then ,

$$(6.2.11) \qquad |a_{mN+1}| \leq \frac{1+\eta}{\dfrac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{N+1}-d_1)|}{(d_{N+1}-d_1)}} \times$$

$$\times \prod_{j=0}^{m-1} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{jN+1}-d_1)|}{d_{((j+1)N+1)}-d_1}$$

for $m = 2, 3, \ldots,$

$$(6.2.12) \, |a_{mN+p}| \leq \frac{2d_1(1-\alpha)\cos\lambda}{d_{mN+p}} \; \frac{1+\eta}{\dfrac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+ (d_{N+1}-d_1)|}{(d_{N+1}-d_1)}} \times$$

$$\times \prod_{j=1}^{m} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{jN+1}-d_1)|}{(d_{jN+1}-d_1)}$$

for $m = 1, 2, \ldots,$ and $p = 2, 3, \ldots, N$, where

$$|a_{N+1}| = \eta \leq \frac{2d_1(1-\alpha)\cos\lambda}{d_{N+1}-d_1} .$$

Proof. From (6.2.7) , we have for $n = 2, 3, \ldots$

$$(6.2.13) \; (d_n-\delta_n d_1)^2 |a_n|^2 \leq 4d_1(1-\alpha)\cos^2\lambda[\, d_1(1-\alpha) + \sum_{k=2}^{n-1} (d_k-\delta d_1)\delta_k |a_k|^2]$$

where,

$$\delta_k = \begin{cases} 1 , & k = NP+1 , \; P = 1, 2, \ldots \\ 0 , & \text{otherwise.} \end{cases}$$

When $n = N+1$, (6.2.13) gives.

$$(6.2.14) \qquad |a_{N+1}| = \eta \leq \frac{2d_1(1-\alpha)\cos\lambda}{d_{N+1}-d_1}.$$

We prove (6.2.11) by applying induction on m.

For $n = 2N+1$, (6.2.13) yields

$$(d_{2N+1}-d_1)^2|a_{2N+1}|^2 \leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+(d_{N+1}-\alpha d_1)|a_{N+1}|^2]$$

In view of (6.2.14) and the above inequality,

$$(d_{2N+1}-d_1)^2|a_{2N+1}|^2 \leq 4d_1^2(1-\alpha)^2\cos^2\lambda(1+\eta)^2.$$

if $d_1(1-\alpha)+\eta^2(d_{N+1}-\alpha d_1) \leq d_1(1-\alpha)(1+\eta)^2$ and the later inequality is certainly true. Thus, (6.2.11) holds for $m = 2$.

Now, assume that (6.2.11) is true for $m = 1, 2, 3, \ldots, (q-1)$. Then, for $n = (qN+1)$, (6.2.13) and the induction hypothesis gives

$$(d_{qN+1}-d_1)^2|a_{qN+1}|^2 \leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+(d_{N+1}-\alpha d_1)\eta^2 + \\ + \sum_{m=2}^{q-1}(d_{mN+1}-\alpha d_1)|a_{mN+1}|^2]$$

$$\leq \frac{(1+\eta)^2(d_{qN+1}-d_1)^2}{\left[\frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{N+1}-d_1)|}{(d_{N+1}-d_1)}\right]^2} \times$$

$$\times \prod_{j=o}^{q-1}\left[\frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{jN+1}-d_1)|}{d_{((j+1)N+1)}-d_1}\right]^2$$

where, the last inequality follows by induction on q. This proves the inequality in (6.2.11) for $m = q$.

Next, we prove (6.2.12) again by using induction on n. When $n = N+p$, $p = 2, \ldots, N$, (6.2.13) and (6.2.14) gives

$$(d_{N+1}-d_1)^2 |a_{N+p}|^2 \leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+(d_{N+1}-d_1)\eta^2]$$
$$\leq 4d_1^2(1-\alpha)^2\cos^2\lambda(1+\eta)^2.$$

The above inequality proves (6.2.12) for $m = 1$.

For $n = qN+p$, the contribution on the right hand side of (6.2.13) comes only from $a_{N+1}, a_{2N+1}, \ldots, a_{(q-1)N+1}$ and hence we use the induction hypothesis as in the proof of (6.2.11) to deduce that

$$|a_{qN+p}| \leq \frac{2d_1(1-\alpha)\cos\lambda(1+\eta)}{d_{qN+p}\dfrac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{N+1}-d_1)|}{(d_{N+1}-d_1)}} \times$$

$$\times \prod_{j=1}^{q} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{jN+1}-d_1)|}{d_{jN+1}-d_1}$$

This proves (62.12) and the proof of Theorem 6.2.2 is complete.

6.3 This section deals with the determination of coefficient bounds of Generalized λ-spirallike functions $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ having order α, $0 \leq \alpha < 1$. The influence of the coefficients $a_{k+1}, \ldots, a_{2k}$ on the boundsof other coefficients of f is also found. For $k = 1$, this reduces to the investigation of influence of fixed second coefficient on the remaining coefficients of f.

Unless otherwise stated, we shall assume throughout in this and the following section that the sequence $\{d_n\}_{n=1}^{\infty}$ of non-decreasing positive numbers in (1.4.1) satisfies

$$(6.3.1) \qquad (d_{qk+1}-d_1) \geq (d_{k+1}-d_2+1)(d_{q+1}-d_1)$$

for $k = 1, 2, \ldots,$ and $q = 1, 2, \ldots$.

We now prove

<u>Theorem</u> 6.3.1 <u>Suppose</u> $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in S_D^k(\lambda, \alpha)$. <u>Then</u>,

$$(6.3.2) \quad |a_n| \leq \frac{(d_{k+1}-d_2+1)(d_{m+1}-d_1)}{(d_n-d_1)} \times$$

$$\times \prod_{j=o}^{m-1} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{j+1}-d_1)|}{d_{j+2}-d_1}$$

<u>where</u>, $mk+1 \leq n \leq (m+1)k$; $m = 1, 2, \ldots,$ <u>and</u> $k = 1, 2, \ldots$.

The following lemmas are needed in the proof of Theorem 6.3.1.

<u>Lemma</u> 6.3.1 <u>If</u> $k = 1, 2, \ldots,$ $q = 1, 2, \ldots,$ <u>and</u> $o \leq \alpha < 1$, <u>then</u>

$$(6.3.3) \quad (d_n-d_1)^2 \geq \frac{(d_n-\alpha d_1)(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2}{d_1(1-\alpha)+(d_{k+1}-d_2+1)(d_{q+1}-d_1)}$$

<u>for</u> $n \geq qk+1$.

<u>Proof</u>. Since, $\{d_n\}_{n=1}^{\infty}$ is non-decreasing, the function $(d_n-d_1)^2/(d_n-\alpha d_1)$ is an increasing function of n. Therefore, in view of (6.3.1), for $n \geq qk+1$, we have

$$\frac{(d_n-d_1)^2}{(d_n-\alpha d_1)} \geq \frac{(d_{qk+1}-d_1)^2}{(d_{qk+1}-\alpha d_1)}$$

$$\geq \frac{(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2}{d_1(1-\alpha)\ +(d_{k+1}-d_2+1)(d_{q+1}-d_1)} \ .$$

This prove (6.3.3).

<u>Lemma</u> 6.3.2 <u>If</u> p <u>and</u> q <u>are positive integers</u>, <u>then for</u> $o \leqq \alpha < 1$, we have

$$(6.3.4)\quad 4d_1(1-\alpha)\cos^2\lambda[\, d_1(1-\alpha)+\sum_{m=1}^{q-1}\{d_1(1-\alpha)+(d_{k+1}-d_2+1)(d_{m+1}-d_1)\}\, x$$

$$x\prod_{j=o}^{m-1}\frac{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}\,]$$

$$\leq (d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2\prod_{j=o}^{q-1}\frac{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}\ .$$

<u>Proof</u>. We shall prove (6.3.4) by induction on q . For q = 1, the left hand side of (6.3.4) gives

$$4d_1^2(1-\alpha)^2\cos^2\lambda = (d_{k+1}-d_2+1)^2(d_2-d_1)^2\,\frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}|^2}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2}$$

and so (6.3.4) is true for q = 1. For q = 2, the left hand side of (6.3.4) becomes

$$4d_1(1-\alpha)\cos^2\lambda[\, d_1(1-\alpha)+\{\, d_1(1-\alpha)+$$

$$+(d_{k+1}-d_2+1)(d_2-d_1)\}\; \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}|^2}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2}\;]$$

$$= 4d_1^2(1-\alpha)^2\cos^2\lambda\ \times$$

$$\times\ [\frac{(d_{k+1}-d_2+1)^2(d_2-d_1)^2+(d_{k+1}-d_2+1)(d_2-d_1)4d_1(1-\alpha)\cos^2\lambda}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2}+$$

$$+\frac{4d_1^2(1-\alpha)^2\cos^2\lambda}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2}\quad]$$

$$= \frac{|2d_1(1-\alpha)\cos^2\lambda e^{-i\lambda}|^2}{(d_{k+1}-d_2+1)(d_2-d_1)^2}\ |2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{k+1}-d_2+1)(d_2-d_1)|^2$$

$$=(d_{k+1}-d_2+1)^2(d_3-d_1)^2\ \prod_{j=0}^{1}\frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}\ .$$

Thus, (6.3.4) is true for $q = 2$.

Now, assume that (6.3.4) is true for $q = 1, 2, \ldots, (n-1)$. Then for $q = n$,

$$4d_1(1-\alpha)\cos^2\lambda[\, d_1(1-\alpha)\ +\ \sum_{m=1}^{n-2}\{\, d_1(1-\alpha)+(d_{k+1}-d_2+1)(d_{m+1}-d_1)\ \}\ \times$$

$$\times\ \prod_{j=0}^{m-1}\frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}+(d_1(1-\alpha)+(d_{k+1}-d_2+1)(d_n-d_1)\times$$

$$\times\ \prod_{j=0}^{n-2}\frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2})]\quad \lneqq$$

$$= (d_{k+1}-d_2+1)^2(d_n-d_1)^2 \prod_{j=o}^{n-2} \frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2} +$$

$$+ 4d_1(1-\alpha)\cos^2\lambda(d_1(1-\alpha)+(d_{k+1}-d_2+1)(d_2-d_1)) \times$$

$$\times \prod_{j=o}^{n-2} \frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}$$

$$=(d_{k+1}-d_2+1)^2(d_{n+1}-d_1)^2 \prod_{j=o}^{n-1} \frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2} .$$

This proves (6.3.4) for $q = n$ and the proof of Lemma 6.3.2 is complete.

Proof of Theorem 6.3.1 . Set ,

$$g(z) = e^{i\lambda} z \frac{Df(z)}{d_1 f(z)} .$$

Since $g(z) \in S_D^k(\lambda,\alpha)$, there exists a bounded regular function $w(z)$ such that $w(o) = o$, $|w(z)| < 1$ and

$$(6.3.5) \qquad w(z) = \frac{g(z)-e^{i\lambda}}{g(z)+e^{-i\lambda}-2\alpha\cos\lambda} = \sum_{n=k}^{\infty} w_n z^n .$$

Equating the coefficients of the same powers of z on both sides of the equation

$$(6.3.6) \qquad e^{i\lambda}\left[\sum_{n=k+1}^{\infty}(d_n-d_1)a_n z^n\right] = \left(\sum_{n=k}^{\infty} w_n z^n\right) \times$$

$$\times\left[2d_1(1-\alpha)z\cos\lambda+\sum_{n=k+1}^{\infty}(e^{i\lambda}d_n+e^{-i\lambda}d_1-2d_1(1-\alpha)\cos\lambda)a_n z^n\right]$$

we obtain ,

(6.3.7) $$e^{i\lambda}(d_n-d_1)a_n = 2d_1(1-\alpha)\cos\lambda\ w_n\ ,\ n = k+1,\ldots,2k.$$

Since $|w(z)| < 1$, it follows that $\sum_{n=k}^{\infty} |w_n|^2 \leq 1$ and therefore

(6.3.8) $$\sum_{n=k+1}^{2k} |w_n|^2 < 1.$$

From (6.3.7) and (6.3.8) , we find that

(6.3.9) $$\sum_{n=k+1}^{2k} (d_n-d_1)^2 \ |a_n|^2 \leq 4d_1^2(1-\alpha)^2 \cos^2\lambda.$$

We rewrite (6.3.6) in the form

(6.3.10) $$\sum_{n=k+1}^{p} e^{i\lambda}(d_n-d_1)a_n z^n + \sum_{n=p+1}^{\infty} c_n z^n = \{2d_1(1-\alpha)z\cos\lambda +$$

$$+ \sum_{n=k+1}^{p-k} (d_n e^{i\lambda}+e^{-i\lambda}d_1 - 2d_1\alpha\cos\lambda)a_n z^n\}\ (\sum_{n=k}^{\infty} w_n z^n)$$

where the constants c_n's occuring in (6.3.10) are determined by the identity in (6.3.5). Since (6.3.10) has the form

$$F(z) = G(z)w(z)$$

where $|w(z)| < 1$, it follows that

(6.3.11) $$\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |G(re^{i\theta})|^2 d\theta$$

for each r , $o < r < 1$. Expressing (6.3.11) in terms of the coefficients in (6.3.10) , we obtain

(6.3.12) $$\sum_{n=k+1}^{p} (d_n-d_1)^2 |a_n|^2 \ r^{2n} + \sum_{n=p+1}^{\infty} |c_n|^2 \ r^{2n}$$

$$\leq 4d_1^2(1-\alpha)^2\cos^2\lambda + \sum_{n=k+1}^{p-k} |d_n e^{i\lambda}+d_1 e^{-i\lambda}-2d_1\alpha\cos\lambda|^2 |a_n|^2 r^{2n}.$$

Making $r \to 1$, we have

$$(6.3.13)\quad \sum_{n=k+1}^{p} (d_n-d_1)^2|a_n|^2 \leq 4d_1^2(1-\alpha)^2\cos^2\lambda + \\ + \sum_{n=k+1}^{p-k} |d_n e^{i\lambda}+d_1(e^{-i\lambda}-2\alpha\cos\lambda)|^2|a_n|^2.$$

Simplifying the quantity under modulus sign on the right side of (6.3.13), the above inequality becomes

$$\sum_{n=k+1}^{p} (d_n-d_1)^2|a_n|^2 \leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+ \\ + \sum_{n=k+1}^{n-k} \{d_1(1-\alpha)+(d_n-\alpha d_1)\}|a_n|^2] + \sum_{n=k+1}^{p-k} (d_n-d_1)^2|a_n|^2.$$

Or,

$$(6.3.14)\quad \sum_{n=p-k+1}^{p} (d_n-d_1)^2|a_n|^2 \leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha) \\ + \sum_{n=k+1}^{p-k} (d_n-\alpha d_1)|a_n|^2].$$

Now using an induction argument and (6.3.14), we establish the inequality

$$(6.3.15)\quad \sum_{n=mk+1}^{(m+1)k} (d_n-d_1)^2|a_n|^2 \leq (d_{k+1}-d_2+1)^2(d_{m+1}-d_1)^2 \times \\ \times \prod_{j=o}^{m-1} \frac{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)} + (d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}$$

from which the desired inequality (6.3.4) follows.

For $m = 1$, (6.3.15) becomes $\sum_{n=k+1}^{2k} (d_n-d_1)^2|a_n|^2 \leq 4d_1^2(1-\alpha)^2\cos^2\lambda$, which is true by (6.3.9). Thus, the inequality (6.3.15) is true for $m = 1$.

Assume that (6.3.15) is true for $m = 1, 2, \ldots, (q-1)$. Using $p = (q+1)k$ in (6.3.14), we obtain

$$\sum_{n=qk+1}^{(q+1)k} (d_n - d_1)^2 |a_n|^2$$

$$\leq 4d_1(1-\alpha)\cos^2\lambda \Big[d_1(1-\alpha) + \sum_{m=1}^{q-1} \Big(\sum_{n=mk+1}^{(m+1)k} (d_n - \alpha d_1)|a_n|^2 \Big) \Big]$$

$$\leq 4d_1(1-\alpha)\cos^2\lambda \Big[d_1(1-\alpha) +$$

$$+ \sum_{m=1}^{q-1} \frac{d_1(1-\alpha) + (d_{k+1} - d_2 + 1)(d_{m+1} - d_1)}{(d_{k+1} - d_2 + 1)^2 (d_{m+1} - d_1)^2} \sum_{n=mk+1}^{(m+1)k} (d_n - d_1)^2 |a_n|^2 \Big]$$

$$\leq 4d_1(1-\alpha)\cos^2\lambda \Big[d_1(1-\alpha) + \sum_{m=1}^{q-1} \{ d_1(1-\alpha) + (d_{k+1} - d_2 + 1)(d_{m+1} - d_1) \} \times$$

$$\times \prod_{j=o}^{m-1} \frac{\left| \dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1} - d_2 + 1} + (d_{j+1} - d_1) \right|^2}{(d_{j+2} - d_1)^2} \Big].$$

The last inequality is obtained from the assumption that (6.3.15) is true for $m = 1, 2, \ldots (q-1)$. Finally, by using Lemma 6.3.2, we obtain

$$\sum_{n=qk+1}^{(q+1)k} (d_n - d_1)^2 |a_n|^2 \leq (d_{k+1} - d_2 + 1)^2 (d_{q+1} - d_1)^2 \times$$

$$\times \prod_{j=o}^{q-1} \frac{\left| \dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1} - d_2 + 1} + (d_{j+1} - d_1) \right|^2}{(d_{j+2} - d_1)^2} .$$

This establishes (6.3.15) for $m = q$. The inequality in (6.3.2) clearly follows from (6.3.15).

For $k = 1$ in Theorem 6.3.1, we get the following :

Corollary 6.3.1 If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_D^1(\lambda,\alpha)$, then

$$(6.3.16) \qquad |a_n| \leq \prod_{j=o}^{n-2} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{j+1}-d_1)|}{d_{j+2}-d_1} .$$

The following coefficient estimates for k-fold symmetric functions also follows from Theorem 6.3.1.

Corollary 6.3.2 If $f(z) = z + \sum_{m=1}^{\infty} a_{mk+1} z^{mk+1} \in S_D^k(\lambda,\alpha)$, then

$$(6.3.17) \qquad |a_{mk+1}| \leq \frac{(d_{k+1}-d_2+1)(d_{m+1}-d_1)}{d_{mk+1}-d_1} \times$$

$$\times \prod_{j=o}^{m-1} \frac{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|}{d_{j+2}-d_1} .$$

Remarks 1. The class $S_D^k(\lambda,\alpha)$ may contain non univalent functions also. For, consider the function $F(z) = z+z^{k+1}$ and $d_n \equiv 1$. Then F is not univalent in Δ. However, $\mathrm{Re}\{e^{i\lambda}zDF(z)/d_1F(z)\} = \cos\lambda > \alpha\cos\lambda$ for $|\lambda| < \pi/2$, $o \leq \alpha < 1$ and $z \in \Delta$, so that $F \in S_D^k(\lambda,\alpha)$.

2. For $d_n \equiv n$, Theorem 6.3.1 gives the result due to Srivastava [101]. For this choice of d_n's Theorem 6.3.1 also include the result of Boyd [9] for $\lambda = o$, MacGregor [52] for $\lambda = o$, $\alpha = o$ and Zamorski [109] for $\alpha = o$ and $k = 1$. Likewise for $d_n \equiv n$, Corollary 6.3.1 gives the results for starlike functions obtained by Goluzin [23]. For $k = 1$, the estimate in (6.3.17) gives the result due to Libera [50].

For $k = 2$, (6.3.17) gives the coefficient bounds for odd Generalized λ-spirallike functions of order α given by

$$|a_{2m+1}| \leq \frac{(d_3-d_2+1)(d_{m+1}-d_1)}{d_{2m+1}-d_1} \times$$

$$\times \prod_{j=o}^{m-1} \frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1} + (d_{j+1}-d_1)\right|}{d_{j+2}-d_1}$$

for $m = 1, 2, \ldots$.

3. The inequality (6.3.2) is sharp for $d_n \equiv n$ and $n = mk+1$ with equality for the function given by

$$F(z) = \frac{z}{(1-z^k)^{2(1-\alpha)\cos\lambda e^{-i\lambda}}}$$

4. For $k = 1$, the inequality (6.3.1) is clearly true. Therefore, all the results obtained in this section hold for any non-decreasing sequence $\{d_n\}_{n=1}^{\infty}$ of positive numbers, if $k=1$.

Our next theorem shows, in particular, how the coefficients $a_{k+1}, \ldots, a_{2k}$ influences the bounds of other coefficients of Generalized λ-spirallike functions.

<u>Theorem</u> 6.3.2 <u>Suppose</u> $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in S_D^k(\lambda,\alpha)$. <u>Then</u>,

$$(6.3.18) \quad |a_n| \le \frac{(d_{q+1}-d_1)(d_{k+1}-d_2+1)\left(1+\frac{\mu}{(d_{k+1}-d_2+1)(d_2-d_1)}\right)}{(d_n-d_1)\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)(d_2-d_1)}+1\right|} \times$$

$$\times \prod_{j=o}^{q-1} \frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|}{d_{j+2}-d_1}$$

<u>for</u> $qk+1 \le n \le (q+1)k$; $q = 2,3,\ldots,$ <u>where</u>

$$\sum_{n=k+1}^{2k} (d_n-d_1)^2 |a_n|^2 = \mu^2 \le |2d_1(1-\alpha)\cos\lambda e^{-i\lambda}|^2 .$$

We need the following Lemma.

<u>Lemma</u> 6.3.3 <u>For each positive integer</u> k <u>and</u> q <u>and for</u> $o \le \alpha < 1$, <u>we have</u>

$$(6.3.19) \quad 4d_1(1-\alpha)\cos^2\lambda\Big[d_1(1-\alpha)+\frac{\{d_1(1-\alpha)+(d_{k+1}-d_2+1)(d_2-d_1)\}\mu^2}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2}+$$

$$+\sum_{m=2}^{q-1}\{d_1(1-\alpha)+(d_{m+1}-d_1)(d_{k+1}-d_2+1)\}\times$$

$$\times\left\{\frac{1+\frac{\mu}{(d_{k+1}-d_2+1)(d_2-d_1)}}{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)(d_2-d_1)}+1\right|}\prod_{j=o}^{m-1}\frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|}{d_{j+2}-d_1}\right\}^2\Big] =$$

$$\leq \frac{(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2}{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)(d_2-d_1)}+1\right|^2}\left(1+\frac{\mu}{(d_{k+1}-d_2+1)(d_2-d_1)}\right)^2 \times$$

$$\times \prod_{j=0}^{q-1} \frac{\left|\frac{2d_1(1-\alpha)\cos\lambda}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2} .$$

Proof. The proof of the lemma follows easily by induction on q and is omitted.

Proof of Theorem 6.3.2. From (6.3.14), we have

$$\sum_{n=p-k+1}^{p} (d_n-d_1)^2|a_n|^2 \leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+\sum_{n=k+1}^{p-k}(d_n-\alpha d_1)|a_n|^2].$$

Putting $p = (q+1)k$ in this inequality, we obtain

(6.3.20) $$\sum_{n=qk+1}^{(q+1)k} (d_n-d_1)\ |a_n|^2$$

$$\leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+\sum_{n=k+1}^{qk}(d_n-\alpha d_1)|a_n|^2]$$

$$\leq 4d_1(1-\alpha)\cos^2\lambda[d_1(1-\alpha)+\sum_{\nu=1}^{q-1}\sum_{n=\nu k+1}^{(\nu+1)k}(d_n-\alpha d_1)|a_n|^2].$$

For $q = 1$, (6.3.20) gives

$$\sum_{n=k+1}^{2k}(d_n-d_1)^2|a_n|^2 = \mu^2 \leq |2d_1(1-\alpha)\cos\lambda\ e^{-i\lambda}|^2 .$$

Now, by using Lemma 6.3.1,

$$\sum_{n=k+1}^{2k} (d_n-\alpha d_1)\,|a_n|^2$$

$$\leq \frac{(d_{k+1}-d_2+1)(d_2-d_1)+d_1(1-\alpha)}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2} \sum_{n=k+1}^{2k} (d_n-d_1)^2|a_n|^2$$

$$= \frac{((d_{k+1}-d_2+1)(d_2-d_1)+d_1(1-\alpha))\mu^2}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2}\,.$$

Using induction, we will prove the inequalities for $q = 1, 2, \ldots$

$$(6.3.21)\quad \sum_{n=qk+1}^{(q+1)k} (d_n-\alpha d_1)\,|a_n|^2$$

$$\leq ((d_{k+1}-d_2+1)(d_2-d_1)+d_1(1-\alpha))\left[\frac{1+\dfrac{\mu}{(d_{k+1}-d_2+1)(d_2-d_1)}}{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)(d_2-d_1)}+1\right|}\right.\times$$

$$\times \left.\prod_{j=o}^{q-1} \frac{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)}+(d_{j+1}-d_1)\right|}{(d_{j+2}-d_1)}\right]^2$$

$$(6.3.22)\quad \sum_{n=qk+1}^{(q+1)k} (d_n-d_1)^2\,|a_n|^2$$

$$\leq \frac{(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2}{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)(d_2-d_1)}+1\right|^2}\left(1+\frac{\mu}{(d_{k+1}-d_2+1)(d_2-d_1)}\right)^2$$

$$\times \prod_{j=o}^{q-1} \frac{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}$$

For $q = 1$, (6.3.21) and (6.3.22) are clearly true. So, assume that (6.3.21) and (6.3.22) are true for $q = 1, 2, \ldots, (m-1)$. Then, for $m = q$, (6.3.20) gives

$$\sum_{n=mk+1}^{(m+1)k} (d_n - d_1)^2 |a_n|^2$$

$$\leq 4d_1(1-\alpha)\cos^2\lambda\Big[d_1(1-\alpha) + \frac{((d_{k+1}-d_2+1)(d_2-d_1)+d_1(1-\alpha))\mu^2}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2} +$$

$$+ \sum_{\nu=2}^{m-1} ((d_{k+1}-d_2+1)(d_{\nu+1}-d_1)+d_1(1-\alpha))\Big\{ \frac{1+\dfrac{\mu}{(d_{k+1}-d_2+1)(d_2-d_1)}}{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)(d_2-d_1)}+1\right|} \times$$

$$\times \prod_{j=0}^{\nu-1} \frac{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)} + (d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)}\Big\}\Big]$$

$$= \frac{(d_{k+1}-d_2+1)^2(d_{m+1}-d_1)^2}{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)(d_2-d_1)}+1\right|^2} \left(1 + \frac{\mu}{(d_{k+1}-d_2+1)(d_2-d_1)}\right)^2 \times$$

$$\times \prod_{j=0}^{m-1} \frac{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1} + d_{j+1}-d_1\right|^2}{(d_{j+2}-d_1)^2}$$

by using Lemma 6.3.3. This proves (6.3.22) for $q = m$. It remains to show that (6.3.21) holds for $m = q$. To this end, we use Lemma 6.3.1 and the above inequality to get

$$\sum_{n=mk+1}^{(m+1)k} (d_n-\alpha d_1)|a_n|^2$$

$$\leq \frac{(d_{m+1}-d_1)(d_{k+1}-d_2+1)+d_1(1-\alpha)}{(d_{m+1}-d_1)^2\,(d_{k+1}-d_2+1)^2} \sum_{n=mk+1}^{(m+1)k} (d_n-d_1)^2|a_n|^2$$

$$\leq ((d_{m+1}-d_1)(d_{k+1}-d_2+1)+d_1(1-\alpha))\ x$$

$$x\ \{\frac{1+\frac{\mu}{(d_{k+1}-d_2+1)(d_2-d_1)}}{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)\,(d_2-d_1)}+1\right|} \prod_{j=o}^{m-1} \frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(d_{j+1}-d_1)\right|}{(d_{j+2}-d_1)}\}^2 .$$

So, (6.3.21) also holds for $q = m$. Thus, we have established the inequalities (6.3.21) and (6.3.22) for $q = 1, 2, \ldots$. Now, the inequality (6.3.18) follows from (6.3.22). This proves the theorem.

<u>Remark</u>. For $d_n \equiv n$ and $k = 1$, a result of Bhatia and Rajasekaran [2] follows from Theorem 6.3.2.

Putting $\mu = o$ in (6.3.18), we have

<u>Corollary</u> 6.3.3 <u>Let</u> $f(z) = z + \sum_{n=2k+1}^{\infty} a_n z^n \in S_D^k(\lambda, \alpha)$ and $\mu = o$. <u>Then</u>,

$$|a_n| \leq \frac{(d_{q+1}-d_1)(d_{k+1}-d_2+1)}{(d_n-d_1)\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)(d_2-d_1)}+1\right|}\ x$$

$$x \prod_{j=o}^{q-1} \frac{\left|\frac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)}+(d_{j+1}-d_1)\right|}{(d_{j+2}-d_1)}$$

<u>for</u> $qk+1 \leq n \leq (q+1)k$, $q = 2, 3, \ldots$.

<u>Corollary</u> 6.3.4 <u>Let</u> $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in S_D^k(\lambda,\alpha)$ <u>and</u> $\mu = 2d_1(1-\alpha)\cos\lambda$. <u>Then</u>,

$$|a_n| \leq \frac{(d_{q+1}-d_1)(d_{k+1}-d_2+1)\left(1+\dfrac{2d_1(1-\alpha)\cos\lambda}{(d_{k+1}-d_2+1)(d_2-d_1)}\right)}{(d_n-d_1)\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_2-d_1)(d_{k+1}-d_2+1)}+1\right|} \times$$

$$\times \prod_{j=o}^{q-1} \frac{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_{k+1}-d_2+1)}+(d_{j+1}-d_1)\right|}{(d_{j+2}-d_1)}$$

for $qk+1 \leq n \leq (q+1)k$, $q = 2, 3, \ldots$.

Putting $k = 1$ in (6.3.18) we have

<u>Corollary</u> 6.3.5 <u>Let</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_D^1(\lambda,\alpha)$. <u>Then</u>

$$|a_n| \leq \frac{\left(1+\dfrac{\mu}{d_2-d_1}\right)}{\left|\dfrac{2d_1(1-\alpha)\cos\lambda e^{-i\lambda}}{(d_2-d_1)}+1\right|} \times \left[\prod_{j=o}^{n-2} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda}+(d_{j+1}-d_1)|}{(d_{j+2}-d_1)}\right]$$

<u>for</u> $n = 2, 3, \ldots$, <u>and where</u> $|a_2| = \mu \leq |2d_1(1-\alpha)\cos\lambda e^{-i\lambda}|$.

Taking $\lambda = o$ and $k = 1$ in (6.3.18), we have

<u>Corollary</u> 6.3.6 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_D^1(o,\alpha)$. Then

$$|a_n| \leq \frac{1+\dfrac{\mu}{d_2-d_1}}{\left(\dfrac{2d_1(1-\alpha)}{d_2-d_1}+1\right)} \prod_{j=o}^{n-2} \frac{2d_1(1-\alpha)+(d_{j+1}-d_1)}{d_{j+2}-d_1} .$$

We also have the following result for k-fold symmetric function whose proof is similar to that of Theorem 6.3.2 and is omitted.

Theorem 6.3.3 Let $f(z) = z + \sum_{m=1}^{\infty} a_{mk+1} z^{mk+1} \in S_D^k(\lambda, \alpha)$. Then,

$$|a_{mk+1}| \leq \frac{1 + \frac{\mu}{d_2 - d_1}}{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda} + (d_{k+1} - d_1)|} \times$$

$$\times \prod_{j=0}^{m-1} \frac{|2d_1(1-\alpha)\cos\lambda e^{-i\lambda} + (d_{jk+1} - d_1)|}{d_{((j+1)k+1)} - d_1}.$$

Remark. Some of the results in [67] follow from our results found in this section, if $d_n \equiv n$.

6.4 In this section, we determine the coefficient estimates for Generalized λ-spirallike function of order α and type β. As in Section 6.3, we assume throughout in this section that the sequence $\{d_n\}_{n=1}^{\infty}$ of non-decreasing positive numbers satisfies (6.3.1).

The following lemmas are needed in the sequel.

Lemma 6.4.1 For $n \geq qk+1$, $k = 1, 2, \ldots$, $q = 1, 2, \ldots$ and $0 < \beta \leq 1$

(6.4.1) $(d_n - d_1)^2 \geq$

$$\geq \frac{(d_{k+1} - d_2 + 1)^2 (d_{q+1} - d_1)^2 [|(2\beta-1)(d_n - d_1) + 2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2 - (d_n - d_1)^2]}{[|(d_{k+1} - d_2 + 1)(d_{q+1} - d_1)(2\beta-1) + 2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2 - (d_{k+1} - d_2 + 1)^2 (d_{q+1} - d_1)^2]}$$

The proof of (6.4.1) is easy though computational and we omit it.

Lemma 6.4.2 If k and q are positive integers, then

$$(6.4.2)\quad 4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda + \sum_{m=1}^{q-1}\{|(2\beta-1)(d_{k+1}-d_2+1)(d_{m+1}-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2 -$$

$$-(d_{k+1}-d_2+1)^2(d_{m+1}-d_1)^2\} \times$$

$$\times \prod_{j=0}^{m-1} \frac{\left|\frac{2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1} + (2\beta-1)(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}$$

$$= (d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2 \times$$

$$\times \prod_{j=0}^{q-1} \frac{\left|\frac{2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1} + (2\beta-1)(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}$$

An induction argument on q establishes (6.4.2) rather easily and we omit the details of its proof.

Theorem 6.4.1 Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in S_D^k(\lambda,\alpha,\beta)$.

(i) If

$$d_1\beta(1-\alpha)((d_{k+1}-d_1)+d_1(1-\alpha))\cos^2\lambda$$

$$> (1-\beta)(d_{k+1}-d_1)((d_{k+1}-d_1) + d_1(1-\alpha)\cos^2\lambda).$$

Then,

$$(6.4.3)\quad |a_n| \le \frac{(d_{k+1}-d_2+1)(d_{m+1}-d_1)}{(d_n-d_1)} \times$$

$$\times \prod_{j=0}^{m-1} \frac{\left|\frac{2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1} +(2\beta-1)(d_{j+1}-d_1)\right|}{(d_{j+2}-d_1)}$$

for $mk+1 \leq n \leq (m+1)k$, $m = 1, 2, \ldots, (M+1)$, where M is given by

$$M = \left[\!\left[\frac{d_1\beta(1-\alpha)((d_{k+1}-d_1)+d_1(1-\alpha))\cos^2\lambda}{(1-\beta)(d_{k+1}-d_1)((d_{k+1}-d_1)+d_1(1-\alpha)\cos^2\lambda)}\right]\!\right]$$

(6.4.4) $$|a_n| \leq \frac{(d_{k+1}-d_2+1)(d_{M+3}-d_1)}{(d_n-d_1)} \times$$

$$\times \prod_{j=0}^{M+1} \frac{\left|\frac{2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1} + (2\beta-1)(d_{j+1}-d_1)\right|}{d_{j+2}-d_1}$$

for $n > (M+2)k$.

(ii) If $d_1\beta(1-\alpha)((d_{k+1}-d_1)+d_1(1-\alpha))\cos^2\lambda$

$$\leq (1-\beta)(d_{k+1}-d_1)((d_{k+1}-d_1) + d_1(1-\alpha)\cos^2\lambda)$$

then,

(6.4.5) $$|a_n| \leq \frac{2d_1\beta(1-\alpha)\cos\lambda}{(d_n-d_1)}.$$

$[\![x]\!]$ denotes the greatest integer less than equal to x.

Proof. Let

$$g(z) = \frac{zDf(z)}{d_1 f(z)}$$

and define

$$h(z) = \frac{g(z)-1}{2\beta(g(z)-1+(1-\alpha)\cos\lambda e^{-i\lambda}) - (g(z)-1)}$$

$$= \sum_{n=k}^{\infty} h_n z^n.$$

for $m = 1, 2, \ldots, M+1$.

The inequality (6.4.3) follows from (6.4.8).

In view of (6.4.6), the inequality (6.4.8) clearly holds for $m = 1$. To show that (6.4.9) is true for $m = 1$, we use Lemma 6.4.1 and (6.4.6) to get

$$\sum_{n=k+1}^{2k} \{ |(2\beta-1)(d_n-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2-(d_n-d_1)^2 \} |a_n|^2$$

$$\leq \left[\frac{|(2\beta-1)(d_{k+1}-d_2+1)(d_2-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2-(d_{k+1}-d_2+1)^2(d_2-d_1)^2}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2}\times\right.$$

$$\left.\times \sum_{n=k+1}^{2k} (d_n-d_1)^2|a_n|^2\right] \leq |2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}|^2 \times$$

$$\times \frac{|(2\beta-1)(d_{k+1}-d_2+1)(d_2-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2-(d_{k+1}-d_2+1)^2(d_2-d_1)^2}{(d_{k+1}-d_2+1)^2(d_2-d_1)^2}$$

which is (6.4.9) for $m = 1$.

Now, suppose that (6.4.8) and (6.4.9) hold for $m = 1, 2, \ldots, (q-1)$. Using (6.4.9) with $p = (q+1)k$ and the inductive hypothesis concerning (6.4.9), we get

$$\sum_{n=qk+1}^{(q+1)k} (d_n-d_1)^2|a_n|^2 \leq 4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda +$$

$$+ \sum_{n=k+1}^{qk} \{|(2\beta-1)(d_n-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2-(d_n-d_1)^2\} |a_n|^2$$

$$\leq 4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda + \sum_{m=1}^{q-1} \sum_{n=mk+1}^{(m+1)k} \{ |(2\beta-1)(d_n-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2 - (d_n-d_1)^2 \}|a_n|^2$$

$$\leq 4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda +$$

$$+\sum_{m=1}^{q-1}\{|(2\beta-1)(d_{k+1}-d_2+1)(d_{m+1}-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2-$$

$$-(d_{k+1}-d_2+1)^2(d_{m+1}-d_1)^2\}\times\{\prod_{j=0}^{m-1}\frac{|\frac{2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(2\beta-1)(d_{j+1}-d_1)|}{d_{j+2}-d_1}\}^2$$

$$=(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2\prod_{j=0}^{q-1}\frac{|\frac{2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(2\beta-1)(d_{j+1}-d_1)|^2}{(d_{j+2}-d_1)^2}.$$

The last equality follows by use of Lemma 6.4.2. This gives the inequality in (6.4.8) when $m = q$.

It remains to prove that (6.4.9) holds for $m = q$. Continuing the same argument that proved (6.4.8) and using Lemma 6.4.1 and (6.4.8) with $m = q$, we obtain

$$\sum_{n=qk+1}^{(q+1)k}\{|(2\beta-1)(d_n-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2-(d_n-d_1)^2\}|a_n|^2$$

$$\leq [\{\frac{|(2\beta-1)(d_{k+1}-d_2+1)(d_{q+1}-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2}{(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2}-$$

$$-\frac{(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2}{(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2}\}\times\sum_{n=qk+1}^{(q+1)k}(d_n-d_1)^2|a_n|^2]\leq$$

$$\leq \left\{\frac{|(2\beta-1)(d_{k+1}-d_2+1)(d_{q+1}-d_1)+2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}|^2-(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2}{(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2}\right\}\times$$

$$(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2\prod_{j=o}^{q-1}\frac{\left|\frac{2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(2\beta-1)(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}$$

$$= \{|(2\beta-1)(d_{k+1}-d_2+1)(d_{q+1}-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2 -$$

$$-(d_{k+1}-d_2+1)^2(d_{q+1}-d_1)^2\}\times\prod_{j=o}^{q-1}\frac{\left|\frac{2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(2\beta-1)(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}$$

This proves (6.4.9) for $m = q$ and hence the proof of (6.4.3) is complete.

To prove (6.4.4), suppose that $n > (M+2)k$. Putting $p = (q+1)k$ in (6.4.7), we have

$$\sum_{n=qk+1}^{(q+1)k}(d_n-d_1)^2|a_n|^2 \leq 4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda$$

$$+\sum_{n=k+1}^{qk}\{|(2\beta-1)(d_n-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2-(d_n-d_1)^2\}|a_n|^2.$$

Hence for $n > (M+2)k$,

$$(d_n-d_1)^2\quad|a_n|^2 \leq 4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda +$$

$$+\sum_{n=k+1}^{(M+2)k}\{|(2\beta-1)(d_n-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2-(d_n-d_1)^2\}|a_n|^2+$$

$$+\sum_{n=(M+2)k+1}^{qk}\{|(2\beta-1)(d_n-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2-(d_n-d_1)^2\}|a_n|^2$$

$$\leq 4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda + \sum_{n=k+1}^{(M+2)k} \{ |(2\beta-1)(d_n-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2 - (d_n-d_1)^2 \}|a_n|^2$$

$$4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda + \sum_{m=1}^{M+1}\sum_{n=mk+1}^{(m+1)k} \{|(2\beta-1)(d_n-d_1)+ 2\beta d_1(1-\alpha)\cos\lambda e^{-i\lambda}|^2 - (d_n-d_1)^2 \}|a_n|^2$$

$$4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda+ \sum_{m=1}^{M+1} \{ |(2\beta-1)(d_{k+1}-d_2+1)(d_{m+1}-d_1)+2d_1\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2$$

$$(d_{k+1}-d_2+1)^2(d_{m+1}-d_1)^2 \}\times \prod_{j=o}^{M-1} \frac{\left|\frac{2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(2\beta-1)(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}$$

$$(d_{k+1}-d_2+1)^2(d_{M+3}-d_1)^2 \quad \prod_{j=o}^{M+1} \frac{\left|\frac{2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(2\beta-1)(d_{j+1}-d_1)\right|^2}{(d_{j+2}-d_1)^2}.$$

ie last equality is obtained by using Lemma 6.4.2.

iis gives

$$|a_n| \leq \frac{(d_{k+1}-d_2+1)(d_{M+3}-d_1)}{(d_n-d_1)} \quad \prod_{j=o}^{M+1} \frac{\left|\frac{2d_1(1-\alpha)\beta\cos\lambda e^{-i\lambda}}{d_{k+1}-d_2+1}+(2\beta-1)(d_{j+1}-d_1)\right|}{(d_{j+2}-d_1)}$$

ich proves (6.4.4).

i) If $d_1\beta(1-\alpha)\cos^2\lambda((d_{k+1}-d_1)+d_1(1-\alpha))$

$$\leq (1-\beta)(d_{k+1}-d_1)((d_{k+1}-d_1) + d_1(1-\alpha)\cos^2\lambda)$$

en (6.4.7) yields

$$\sum_{n=p-k+1}^{p} (d_n-d_1)^2|a_n|^2 \leq 4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda$$

or, if $n \geqq k+1$,

$$(d_n-d_1)^2 |a_n|^2 \leqq 4d_1^2\beta^2(1-\alpha)^2\cos^2\lambda$$

so that

$$|a_n| \leqq \frac{2d_1\beta(1-\alpha)\cos\lambda}{(d_n-d_1)} .$$

This proves the estimate (6.4.5) and completes the proof of part (ii) of the theorem.

Remarks For $d_n \equiv n$ and $\lambda = o$, Theorem 6.4.1 gives a result in [57]. Further, for $d_n \equiv n$ and $k = 1$, a result in [40] follows from Theorem 6.4.1.

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